PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Theoretical Studies in Nuclear Structure. IV. Wave Functions for the Nuclear p-Shell. Part B. $\langle p^{n}|p^{n-2}p^{2}\rangle$ Fractional Parentage Coefficients

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THEORETICAL STUDIES IN NUCLEAR STRUCTURE
IV. WAVE FUNCTIONS FOR THE NUCLEAR p-SHELL

Part B. $\langle p^n | p^{n-2} p^2 \rangle$ FRACTIONAL PARENTAGE COEFFICIENTS

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The totally antisymmetric states for the p-shell nuclei previously given (part IVA, Jahn & van Wieringen 1951) are transformed into the form of linear combinations of antisymmetric states for $n-2$ particles vector-coupled to antisymmetric states of particles n and $n-1$. The coefficients of the combinations, the $\langle n | n-2, 2 \rangle$ fractional parentage coefficients, are shown to be products of weight factors, orbital factors and charge-spin factors which are tabulated separately, the tabulation being simplified by use of the special unitary group. This form of the wave functions is the most suitable for the evaluation of any two-body interaction and has been used in a series of investigations (J. P. Elliott 1952, 1953; Robinson 1953) relating to the effect of tensor and spin-orbit forces on the levels of p-shell nuclei. The complete p-shell charge-symmetric central force matrix given by Racah (1950) with unspecified phases is recalculated with the standard (amended) phases of part IVA, with the help of the coefficients here tabulated.

Introduction

In a previous paper (Jahn & van Wieringen (1951), referred to in the following as part A), a method was described for constructing the totally antisymmetric states of the p-shell nuclei. The wave functions were there given in the form of linear combinations of totally antisymmetric states of $n-1$ particles vector-coupled to the states of the nth particle, and with phases‡ so chosen that the component orbital parts of the wave function, when the total wave function is resolved in the usual manner into a linear combination of products of

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‡ *Correction to part A*. Owing to an oversight, the orbital coefficients as printed in Jahn & van Wieringen (1951), although giving rise to correctly antisymmetrized wave functions, are not such as to make $x=0$ throughout in the generalized Racah reciprocal relation as stated at the bottom of p. 518. To make x always zero requires the following overall phase changes:

Change the overall sign of the following orbital functions:

$$[\bar{2}]\ S, D; \quad [\bar{3}]\ P, F; \quad [\bar{4}]\ S, D, G; \quad [\bar{1}\bar{3}]\ P, D, F;$$

or, in full, change the sign of the orbital states

$$\left.\begin{array}{r} p^4\ [22] \\ p^7\ [331] \\ p^{10}\ [442] \end{array}\right\} S, D; \quad \left.\begin{array}{r} p^6\ [33] \\ p^9\ [441] \end{array}\right\} P, F; \quad p^8\ [44]\ S, D, G; \quad \left.\begin{array}{r} p^5\ [32] \\ p^8\ [431] \end{array}\right\} P, D, F.$$

These changes do not affect the tabulated matrices of $U^{(2)}$, nor, being overall phase changes, do they affect the standard Young-Yamanouchi phases. In the present paper, when we refer to the standard phases of part A, we assume that they have been amended in the manner explained above. The authors are indebted to Professor G. Racah for calling their attention to the necessity for these changes.

orbital and charge-spin functions, transform under permutations according to the standard orthogonal Young-Yamanouchi representation. In the present paper we bring the same wave functions with the same (amended) phases (see footnote) in the form of linear combinations of totally antisymmetric states of $n-2$ particles vector-coupled to the states of particles n and $n-1$. This form of the wave functions is most suitable for the evaluation of the matrix elements of any two-particle operator, and these wave functions have been used in this form in a series of investigations begun by J. P. Elliott and being continued at Southampton by other investigators on the effect of non-central forces on the levels of p-shell nuclei.

In § 1 a method is described of calculating the orbital $\langle p^n \,|\, p^{n-2}p^2\rangle$ coefficients from the $\langle p^n \,|\, p^{n-1}p\rangle$ and $\langle p^{n-1} \,|\, p^{n-2}p\rangle$ coefficients listed in part A, the method being described in detail for the p^6 states of symmetry [42]. In § 2 the similar considerations for the charge-spin functions are described. In § 3 the method of constructing the totally antisymmetric wave functions is described and proof given that the total $\langle n \,|\, n-2, 2\rangle$ fractional parentage coefficients are direct products of weight coefficients (equal to the square root of ratios of dimensions of representations of permutation groups), orbital coefficients and charge-spin coefficients. Tables 1, 2a and 2b give the values of the Racah coefficients required in the calculation. In tables 3, 4, 5a and 5b the three sets of weight, orbital and charge-spin coefficients are tabulated separately. In this tabulation use is made of symbols describing the irreducible representations of the special unitary groups to which the states belong, this enabling us to avoid repetition, as the same representations occur for varying numbers of particles. A convenient new notation is introduced for these symbols. In the case of the group SU_3 of special unitary transformations in the space of the orbital states of a single p-particle, we use the symbol $(g_1, g_2) \equiv (f_1 - f_2, f_2 - f_3)$, where $[f_1 f_2 f_3]$ is the partition describing the permutation symmetry or representation of the unitary group U_3. In the case of the group SU_4 of special unitary transformations in the four-dimensional space of the charge-spin states of a single particle we use the symbol $(g_1 g_2 g_3) \equiv (f_1 - f_2, f_2 - f_3, f_3 - f_4)$, where $[f_1 f_2 f_3 f_4]$ describes both the permutation symmetry and transformation properties with respect to the unitary group U_4. With this notation the symbol of a representation contragredient to a given representation is obtained by reversing the order of the numbers in the symbol: $(g_2 g_1)$ is contragredient to $(g_1 g_2)$ and $(g_3 g_2 g_1)$ is contragredient to $(g_1 g_2 g_3)$. § 4 is devoted to the redetermination of the complete matrix for a general charge-symmetric central force interaction in the p-shell with the standard phases of this paper. Table 6 lists all the non-vanishing matrix elements of the two-particle interaction $X = \sum_{i<j} X_{ij}$ already tabulated with unspecified phases by Racah (1950). Here

$$X_{ij} = \tfrac{15}{16}\{(u_i^{(1)} \,.\, u_j^{(1)}) - (u_i^{(2)} \,.\, u_j^{(2)})\}\{(\vec{\tau}_i \,.\, \vec{\tau}_j) - (\vec{\sigma}_i \,.\, \vec{\sigma}_j)\}$$

is a two-particle operator of symmetry type (22) (020) ^{11}S, where the first bracket is the symbol $(g_1 g_2)$ of a representation of SU_3 and the second bracket a symbol $(g_1 g_2 g_3)$ of a representation of SU_4. It was shown by Racah that the interaction X is the only charge-symmetric central force interaction whose matrix elements require special calculation. They are calculated here using $\langle n \,|\, n-2, 2\rangle$ parentage coefficients; Racah does not give details of his method of calculation, but presumably he used $\langle n \,|\, n-1, 1\rangle$ coefficients, in which case the calculation is longer (once the coefficients are known).

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1. Calculation of the orbital $\langle p^n | p^{n-2}p^2 \rangle$ coefficients from the $\langle p^n | p^{n-1}p \rangle$ and $\langle p^{n-1} | p^{n-2}p \rangle$ orbital coefficients

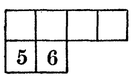
As an example, we begin by describing in detail the calculation for the orbital states $\Phi(p^6[42]\,\kappa LM \,|\, r_6 r_5 \ldots r_1)$ of the p^6 configuration (the symbol κ is introduced merely to distinguish the two D states which occur in the representation $\mathscr{H}_{[42]}$ of U_3). We make use of the following coefficients tabulated in part A:

$$C^{[42]\,\kappa L}_{[41]\,L'[1]},\quad C^{[41]\,L}_{[4]\,L''[1]},\quad C^{[41]\,L'}_{[31]\,L''[1]},$$
$$C^{[42]\,\kappa L}_{[32]\,L'[1]},\quad C^{[32]\,L'}_{[31]\,L''[1]},\quad C^{[32]\,L'}_{[22]\,L''[1]}.$$

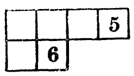
(The notation is again modified: $C^{[42]\,\kappa L}_{[41]\,L'[1]}$ would in the notation of part A be denoted by $\langle p^6[42]\,\kappa L \,|\, p^5[41]\,L' \rangle$; in the notation of Racah (1943) it would be denoted by

$$(p^5([41]\,L'), p, L \,|\}\, p^6[42]\,\kappa L).$$

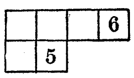
Corresponding to the four possible positions (indicated below) of the numbers 5 and 6 in the regular Young tableau belonging to the partition [42], we have the following four types of expansion:

(A) 

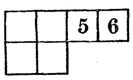
$$\Phi(p^6[42]\,\kappa LM_L \,|\, 22r_4r_3r_2r_1) = \sum_{L'L''} C^{[42]\,\kappa L}_{[41]\,L'[1]}\, C^{[41]\,L}_{[4]\,L''[1]}\, \Phi(p^4[4]\,L''(r_4r_3r_2r_1), p_5, L', p_6, LM_L), \quad (1\,\text{A})$$

(B) 

$$\Phi(p^6[42]\,\kappa LM_L \,|\, 21r_4r_3r_2r_1) = \sum_{L'L''} C^{[42]\,\kappa L}_{[41]\,L'[1]}\, C^{[41]\,L'}_{[31]\,L''[1]}\, \Phi(p^4[31]\,L''(r_4r_3r_2r_1), p_5, L', p_6, LM_L), \quad (1\,\text{B})$$

(C) 

$$\Phi(p^6[42]\,\kappa LM_L \,|\, 12r_4r_3r_2r_1) = \sum_{L'L''} C^{[42]\,\kappa L}_{[32]\,L'[1]}\, C^{[32]\,L'}_{[31]\,L''[1]}\, \Phi(p^4[31]\,L''(r_4r_3r_2r_1), p_5, L', p_6, LM_L), \quad (1\,\text{C})$$

(D) 

$$\Phi(p^6[42]\,\kappa LM_L \,|\, 11r_4r_3r_2r_1) = \sum_{L'L''} C^{[42]\,\kappa L}_{[32]\,L'[1]}\, C^{[32]\,L'}_{[22]\,L''[1]}\, \Phi(p^4[22]\,L''(r_4r_3r_2r_1), p_5, L', p_6, LM_L). \quad (1\,\text{D})$$

Here $(r_4r_3r_2r_1)$ is any Yamanouchi symbol (see part A), consistent with the 4-particle partition: e.g. r_4 is the number of the row (in the above, the first or second row) of the tableau in which the number 4 appears.

We may transform the states occurring on the right to states in which the orbital states of particles 5 and 6 (denoted by p_5 and p_6) are vector-coupled together, by means of the general transformation

$$\Phi(l^{n-2}[f'']\,\kappa''L''(r_{n-2}r_{n-1}\ldots r_2r_1), l_{n-1}, L', l_n, LM_L)$$
$$= \sum_{L_2} U(L''lLl; L'L_2)\, \Phi(l^{n-2}[f'']\,\kappa''L''(r_{n-2}r_{n-1}\ldots r_2r_1), l_{n-1}l_nL_2, LM_L). \quad (2)$$

The values of the Racah coefficients $U(L''1L1; L'L_2)$ required for the whole of the nuclear p-shell are tabulated in table 1.†

† The tables appear at the end of the paper.

We have then in the four cases

(A) $\Phi(p^6[42]\,\kappa LM_L\,|\,22r_4r_3r_2r_1)$

$$=\sum_{L''L_2}\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[41]\,L'[1]}\, C^{[41]\,L'}_{[4]\,L''[1]}\, \Phi(p^4[4]\,L''(r_4r_3r_2r_1),p_5p_6L_2,LM_L), \quad (3\text{A})$$

(B) $\Phi(p^6[42]\,\kappa LM_L\,|\,21r_4r_3r_2r_1)$

$$=\sum_{L''L_2}\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[41]\,L'[1]}\, C^{[41]\,L'}_{[31]\,L''[1]}\, \Phi(p^4[31]\,L''(r_4r_3r_2r_1),p_5p_6L_2,LM_L), \quad (3\text{B})$$

(C) $\Phi(p^6[42]\,\kappa LM_L\,|\,12r_4r_3r_2r_1)$

$$=\sum_{L''L_2}\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[32]\,L'[1]}\, C^{[32]\,L'}_{[31]\,L''[1]}\, \Phi(p^4[31]\,L''(r_4r_3r_2r_1),p_5p_6L_2,LM_L), \quad (3\text{C})$$

(D) $\Phi(p^6[42]\,\kappa LM_L\,|\,11r_4r_3r_2r_1)$

$$=\sum_{L''L_2}\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[32]\,L'[1]}\, C^{[32]\,L'}_{[22]\,L''[1]}\, \Phi(p^4[22]\,L''(r_4r_3r_2r_1),p_5p_6L_2,LM_L). \quad (3\text{D})$$

The functions A and D are symmetrical in particles 5 and 6, so that in the above expansions in these cases L_2 is even, taking on the values appropriate to the representation $\mathscr{H}_{[2]}$ of the unitary group U_3. It follows that we may write the expansions (3A) and (3D) as

(A) $\Phi(p^6[42]\,\kappa LM_L\,|\,22r_4r_3r_2r_1)$

$$=\sum_{L''L_2} C^{[42]\,\kappa L}_{[4]\,L''[2]\,L_2}\, \Phi(p^4[4]\,L''(r_4r_3r_2r_1),p_5p_6[2]\,L_2,LM_L), \quad (4\text{A})$$

(D) $\Phi(p^6[42]\,\kappa LM_L\,|\,11r_4r_3r_2r_1)$

$$=\sum_{L''L_2} C^{[42]\,\kappa L}_{[22]\,L''[2]\,L_2}\, \Phi(p^4[22]\,L''(r_4r_3r_2r_1),p_5p_6[2]\,L_2,LM_L), \quad (4\text{D})$$

with
$$C^{[42]\,\kappa L}_{[4]\,L''[2]\,L_2}=\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[41]\,L'[1]}\, C^{[41]\,L'}_{[4]\,L''[1]}, \quad (5)$$

and
$$C^{[42]\,\kappa L}_{[22]\,L''[2]\,L_2}=\sum_{L'} U(L''1L1;L'L_2)\, C^{[42]\,\kappa L}_{[32]\,L'[1]}\, C^{[32]\,L'}_{[22]\,L''[1]}. \quad (6)$$

These coefficients, taken together with the appropriate Wigner coefficients, then determine the $[42]\,\kappa LM_L$ component of the product representations $\mathscr{H}_{[4]}\times\mathscr{H}_{[2]}$ and $\mathscr{H}_{[22]}\times\mathscr{H}_{[2]}$ of the group U_3.

The functions (B) and (C) are neither symmetrical nor antisymmetrical in particles 5 and 6, so that L_2 takes on both even and odd values appropriate to both the representations $\mathscr{H}_{[2]}$ and $\mathscr{H}_{[11]}$ of U_3. Since we know, however, the matrix of P_{56} for these functions, from the form of the Young-Yamanouchi standard orthogonal representation, we may construct from them two linear combinations which are respectively symmetrical and antisymmetrical with respect to P_{56}, and which belong then respectively to the representations $\mathscr{H}_{[2]}$ and $\mathscr{H}_{[11]}$.

We have, in fact, in the general case (cf. part A), with $r<s$ and $\mu=f_r-f_s+(s-r)$,

$$\left.\begin{aligned} P_{n-1,\,n}\,\phi(rsr_{n-2}\dots r_2r_1) &= \frac{1}{\mu}\,\phi(rsr_{n-2}\dots r_2r_1)+\frac{\sqrt{(\mu^2-1)}}{\mu}\,\phi(srr_{n-2}\dots r_2r_1),\\ P_{n-1,\,n}\,\phi(srr_{n-2}\dots r_2r_1) &= \frac{\sqrt{(\mu^2-1)}}{\mu}\,\phi(rsr_{n-2}\dots r_2r_1)-\frac{1}{\mu}\,\phi(srr_{n-2}\dots r_2r_1).\end{aligned}\right\} \quad (7)$$

Hence, with

$$\left.\begin{aligned}\phi([rs]\,r_{n-2}\ldots r_2r_1) &= \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\,\phi(rsr_{n-2}\ldots r_2r_1)+\sqrt{\left(\frac{\mu-1}{2\mu}\right)}\,\phi(srr_{n-2}\ldots r_2r_1),\\ \phi(\{rs\}\,r_{n-2}\ldots r_2r_1) &= \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\,\phi(rsr_{n-2}\ldots r_2r_1)-\sqrt{\left(\frac{\mu+1}{2\mu}\right)}\,\phi(srr_{n-2}\ldots r_2r_1),\end{aligned}\right\} \tag{8}$$

we have

$$\left.\begin{aligned}P_{n-1,\,n}\,\phi([rs]\,r_{n-2}\ldots r_2r_1) &= +\phi([rs]\,r_{n-2}\ldots r_2r_1),\\ P_{n-1,\,n}\,\phi(\{rs\}\,r_{n-2}\ldots r_2r_1) &= -\phi(\{rs\}\,r_{n-2}\ldots r_2r_1).\end{aligned}\right\} \tag{9}$$

Functions chosen thus may be referred to as belonging to the diagonalized Young-Yamanouchi-Rutherford representation. There is, of course, an arbitrary phase choice for the two functions; in what follows we adhere to the foregoing choice and may refer to the functions so defined as forming the basis of the standard diagonalized Young-Yamanouchi representation. Since the transformation is orthogonal, we may also write

$$\left.\begin{aligned}\phi(rsr_{n-2}\ldots r_2r_1) &= \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\,\phi([rs]\,r_{n-2}\ldots r_2r_1)+\sqrt{\left(\frac{\mu-1}{2\mu}\right)}\,\phi(\{rs\}\,r_{n-2}\ldots r_2r_1),\\ \phi(srr_{n-2}\ldots r_2r_1) &= \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\,\phi([rs]\,r_{n-2}\ldots r_2r_1)-\sqrt{\left(\frac{\mu+1}{2\mu}\right)}\,\phi(\{rs\}\,r_{n-2}\ldots r_2r_1).\end{aligned}\right\} \tag{10}$$

In our special case we may hence write the functions (3 B) and (3 C) as follows:

(B) $\Phi(p^6[42]\,\kappa LM_L\,|\,21r_4r_3r_2r_1)$

$$\begin{aligned}&= \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\sum_{L''L_2} C^{[42]\kappa L}_{[31]L''[2]L_2}\,\Phi(p^4[31]\,L''(r_4r_3r_2r_1),\,p_5p_6[2]\,L_2,\,LM_L)\\ &\quad-\sqrt{\left(\frac{\mu+1}{2\mu}\right)}\sum_{L''L_2} C^{[42]\kappa L}_{[31]L''[11]L_2}\,\Phi(p^4[31]\,L''(r_4r_3r_2r_1),\,p_5p_6[11]\,L_2,\,LM_L),\end{aligned} \tag{4 B}$$

(C) $\Phi(p^6[42]\,\kappa LM_L\,|\,12r_4r_3r_2r_1)$

$$\begin{aligned}&= \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\sum_{L''L_2} C^{[42]\kappa L}_{[31]L''[2]L_2}\,\Phi(p^4[31]\,L''(r_4r_3r_2r_1),\,p_5p_6[2]\,L_2,\,LM_L)\\ &\quad+\sqrt{\left(\frac{\mu-1}{2\mu}\right)}\sum_{L''L_2} C^{[42]\kappa L}_{[31]L''[11]L_2}\,\Phi(p^4[31]\,L''(r_4r_3r_2r_1),\,p_5p_6[11]\,L_2,\,LM_L).\end{aligned} \tag{4 C}$$

Comparison with the previous expansions (3 B) and (3 C) gives us the relations

$$C^{[42]\kappa L}_{[31]L''[2]L_2} = \sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{L'} U(L''1L1;L'L_2)\,C^{[42]\kappa L}_{[41]L'[1]}\,C^{[41]L'}_{[31]L''[1]} \tag{11 a}$$

$$= \sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{L'} U(L''1L1;L'L_2)\,C^{[42]\kappa L}_{[32]L'[1]}\,C^{[32]L'}_{[31]L''[1]}, \tag{11 b}$$

$$C^{[42]\kappa L}_{[31]L''[11]L_2} = -\sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{L'} U(L''1L1;L'L_2)\,C^{[42]\kappa L}_{[41]L'[1]}\,C^{[41]L'}_{[31]L''[1]} \tag{12 a}$$

$$= \sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{L'} U(L''1L1;L'L_2)\,C^{[42]\kappa L}_{[32]L'[1]}\,C^{[32]L'}_{[31]L''[1]}, \tag{12 b}$$

where in this particular example μ is the axial distance from 6 to 5 in the scheme $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & 6\\ \hline \; & 5 \\ \cline{1-2}\end{array}$ and is equal to 3.

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Since the reduction of product representations of the unitary group determines simultaneously the reduction of the product of the corresponding representations of the special unitary group, the above formulae hold without change for the special unitary group. If $[f_1 f_2 f_3]$ is the partition associated with the representation $\mathscr{H}_{[f]}$ of U_3, then we use, as stated in the Introduction, $(g_1 g_2) = (f_1 - f_2, f_2 - f_3)$ to distinguish the corresponding representation $\mathscr{K}_{(g)}$ of SU_3. The general result may then be written as

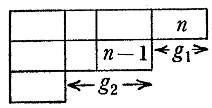
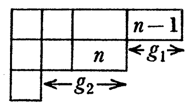
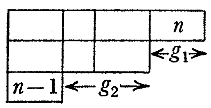
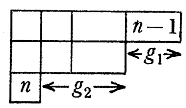
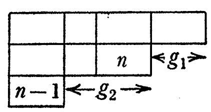
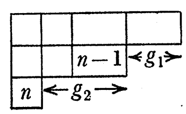
$$C^{(g)\kappa L}_{(g'')\kappa'' L''(20)\,L_2} = \sqrt{\left(\frac{2\mu}{\mu+1}\right)} \sum_{L'\kappa'} U(L''1L1;L'L_2)\, C^{(g)\kappa L}_{(g'_a)\kappa' L'(10)}\, C^{(g'_a)\kappa' L'}_{(g'')\kappa'' L''(10)} \qquad (13a)$$

$$= \sqrt{\left(\frac{2\mu}{\mu-1}\right)} \sum_{L'\kappa'} U(L''1L1;L'L_2)\, C^{(g)\kappa L}_{(g'_b)\kappa' L'(10)}\, C^{(g'_b)\kappa' L'}_{(g'')\kappa'' L''(10)}, \qquad (13b)$$

$$C^{(g)\kappa L}_{(g'')\kappa'' L''(01)\,L_2} = \sqrt{\left(\frac{2\mu}{\mu-1}\right)} \sum_{L'\kappa'} U(L''1L1;L'L_2)\, C^{(g)\kappa L}_{(g'_a)\kappa' L'(10)}\, C^{(g'_a)\kappa' L'}_{(g'')\kappa'' L''(10)} \qquad (14a)$$

$$= -\sqrt{\left(\frac{2\mu}{\mu+1}\right)} \sum_{L'\kappa'} U(L''1L1;L'L_2)\, C^{(g)\kappa L}_{(g'_b)\kappa' L'(10)}\, C^{(g'_b)\kappa' L'}_{(g'')\kappa'' L''(10)}, \qquad (14b)$$

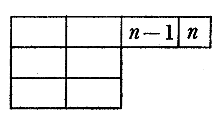
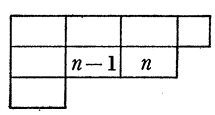
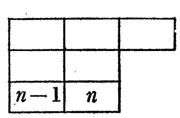
holding for the following sets of values (g), (g''), (g'_a), (g'_b) with the appropriate value of μ indicated and corresponding, as shown, to various positions of the numbers n and $n-1$ in the symmetry scheme:

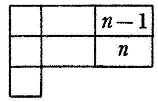
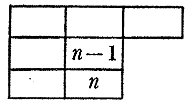
(g)	(g'')	(g'_a)	(g'_b)	μ	(a)	(b)
$(g_1 g_2)$	(g_1, g_2-1)	(g_1-1, g_2)	(g_1+1, g_2-1)	g_1+1		
	(g_1-1, g_2+1)	(g_1-1, g_2)	(g_1, g_2+1)	g_1+g_2+2		
	(g_1+1, g_2)	(g_1+1, g_2-1)	(g_1, g_2+1)	g_2+1		

In addition, the simple relation

$$C^{(g)\kappa L}_{(g'')\kappa'' L''(G)\,L_2} = \sum_{L'\kappa'} U(L''1L1;L'L_2)\, C^{(g)\kappa L}_{(g')\kappa' L'(10)}\, C^{(g')\kappa' L'}_{(g'')\kappa'' L''(10)} \qquad (15)$$

holds for the following cases:

(G)	(g)	(g'')	(g')	
(20)	$(g_1 g_2)$	(g_1-2, g_2)	(g_1-1, g_2)	
		(g_1+2, g_2-2)	(g_1+1, g_2-1)	
		(g_1, g_2+2)	(g_1, g_2+1)	

(G)	(g)	(g'')	(g')	
(01)	$(0g_2)$	$(0, g_2-1)$	$(1, g_2-1)$	
	$(g_1 0)$	$(g_1+1, 0)$	$(g_1, 1)$	

Coefficients for the whole of the nuclear p-shell satisfying these relations are given in the table 4 of $\langle p^n \mid p^{n-2}p^2\rangle$ orbital coefficients.

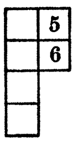
2. Calculation of the charge-spin $\langle \gamma^n \mid \gamma^{n-2}\gamma^2\rangle$ coefficients from the $\langle \gamma^n \mid \gamma^{n-1}\gamma\rangle$ and $\langle \gamma^{n-1} \mid \gamma^{n-2}\gamma\rangle$ coefficients

We begin again by describing in detail, as an example, the calculation for the 6-particle charge-spin states $\Gamma(\gamma^6[\widetilde{42}]\; TSM_TM_S \mid \tilde{r}_6\tilde{r}_5 \ldots \tilde{r}_1)$. We use the following coefficients tabulated in part A:

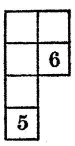
$$C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]},\quad C^{[\widetilde{41}]\,T'S'}_{[\tilde{4}]\,T''S''[1]},\quad C^{[\widetilde{41}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]},$$
$$C^{[\widetilde{42}]\,TS}_{[\widetilde{32}]\,T'S'[1]},\quad C^{[\widetilde{32}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]},\quad C^{[\widetilde{32}]\,T'S'}_{[\widetilde{22}]\,T''S''[1]}.$$

(In the notation of part A, $C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]} \equiv \langle \gamma^6\,[\widetilde{42}]\; TS \mid \gamma^5[\widetilde{41}]\; T'S'\rangle$.)

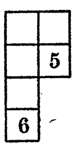
We have then, corresponding to the four possible positions of the numbers 5 and 6 in the standard Young tableau, the following four types of expansion:

$(\tilde{A})$ 

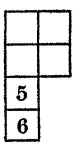
$$\Gamma(\gamma^6[\widetilde{42}]\; TSM_TM_S \mid \tilde{2}\,\tilde{2}\,\tilde{r}_4 \ldots \tilde{r}_1) = \sum_{T'S'T''S''} C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]}\, C^{[\widetilde{41}]\,T'S'}_{[\tilde{4}]\,T''S''[1]}\, \Gamma(\gamma^4[\tilde{4}]\; T''S''(\tilde{r}_4 \ldots \tilde{r}_1), \gamma_5, T'S', \gamma_6, TSM_TM_S), \tag{16A}$$

$(\tilde{B})$ 

$$\Gamma(\gamma^6[\widetilde{42}]\; TSM_TM_S \mid \tilde{2}\,\tilde{1}\,\tilde{r}_4 \ldots \tilde{r}_1) = \sum_{T'S'T''S''} C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]}\, C^{[\widetilde{41}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]}\, \Gamma(\gamma^4[\widetilde{31}]\; T''S''(\tilde{r}_4 \ldots \tilde{r}_1), \gamma_5, T'S', \gamma_6, TSM_TM_S), \tag{16B}$$

$(\tilde{C})$ 

$$\Gamma(\gamma^6[\widetilde{42}]\; TSM_TM_S \mid \tilde{1}\,\tilde{2}\,\tilde{r}_4 \ldots \tilde{r}_1) = \sum_{T'S'T''S''} C^{[\widetilde{42}]\,TS}_{[\widetilde{32}]\,T'S'[1]}\, C^{[\widetilde{32}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]}\, \Gamma(\gamma^4[\widetilde{31}]\; T''S''(\tilde{r}_4 \ldots \tilde{r}_1), \gamma_5, T'S', \gamma_6, TSM_TM_S), \tag{16C}$$

$(\tilde{D})$ 

$$\Gamma(\gamma^6[\widetilde{42}]\; TSM_TM_S \mid \tilde{1}\,\tilde{1}\,\tilde{r}_4 \ldots \tilde{r}_1) = \sum_{T'S'T''S''} C^{[\widetilde{42}]\,TS}_{[\widetilde{32}]\,T'S'[1]}\, C^{[\widetilde{32}]\,T'S'}_{[\widetilde{22}]\,T''S''[1]}\, \Gamma(\gamma^4[\widetilde{22}]\; T''S''(\tilde{r}_4 \ldots \tilde{r}_1), \gamma_5, T'S', \gamma_6, TSM_TM_S), \tag{16D}$$

where $(\tilde{r}_4\tilde{r}_3\tilde{r}_2\tilde{r}_1)$ is any Yamanouchi symbol consistent with the 4-particle partition. As in part A, the tildes are used to denote that we are dealing here with the standard adjoint Yamanouchi representation, e.g. $\tilde{r}_4$ is now the number of the column (first or second column in our case) of the tableau in which the number 4 appears.

We may transform the states occurring on the right by means of the transformation

$$\Gamma(\gamma^{n-2}[\tilde{f}''] \, T''S''(r_{n-2}\ldots r_2 r_1), \gamma_{n-1}, T'S', \gamma_n, TSM_T M_S)$$
$$= \sum_{T_2 S_2} U(T''\tfrac{1}{2}T\tfrac{1}{2}; T'T_2)\, U(S''\tfrac{1}{2}S\tfrac{1}{2}; S'S_2)\, \Gamma(\gamma^{n-2}[\tilde{f}'']\, T''S''(r_{n-2}\ldots r_2 r_1), \gamma_{n-1}\gamma_n\, T_2 S_2, TSM_T M_S), \quad (17)$$

The coefficients of this transformation for the cases occurring in the nuclear p-shell are tabulated in tables $2a$ (n even) and $2b$ (n odd).

Here functions ($\tilde{\text{A}}$) and ($\tilde{\text{D}}$) are antisymmetrical in particles 5 and 6, so that only those values of $T_2 S_2$ occur that are appropriate to the representation $\mathscr{H}_{[\tilde{2}]}$ of U_4. We may hence write in cases ($\tilde{\text{A}}$) and ($\tilde{\text{D}}$):

$$(\tilde{\text{A}}): \quad \Gamma(\gamma^6[\widetilde{42}]\, TSM_T M_S \,|\, \tilde{2}\,\tilde{2}\,\tilde{r}_4\ldots\tilde{r}_1)$$
$$= \sum_{T''S''T_2 S_2} C^{[\widetilde{42}]\,TS}_{[\tilde{4}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(\gamma^4[\tilde{4}]\, T''S''(\tilde{r}_4\ldots\tilde{r}_1), \gamma_5\gamma_6[\tilde{2}]\, T_2S_2, TSM_T M_S), \quad (18\text{A})$$

$$(\tilde{\text{D}}): \quad \Gamma(\gamma^6[\widetilde{42}]\, TSM_T M_S \,|\, \tilde{1}\,\tilde{1}\,\tilde{r}_4\ldots\tilde{r}_1)$$
$$= \sum_{T''S''T_2 S_2} C^{[\widetilde{42}]\,TS}_{[22]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(\gamma^4[\widetilde{22}]\, T''S''(\tilde{r}_4\ldots\tilde{r}_1), \gamma_5\gamma_6[\tilde{2}]\, T_2S_2, TSM_T M_S), \quad (18\text{D})$$

with $$C^{[\widetilde{42}]\,TS}_{[\tilde{4}]\,T''S''[\tilde{2}]\,T_2S_2} = \sum_{T'S'} U(T''\tfrac{1}{2}T\tfrac{1}{2}, T'T_2)\, U(S''\tfrac{1}{2}S\tfrac{1}{2}; S'S_2)\, C^{[\widetilde{42}]\,TS}_{[\tilde{4}]\,T'S'[1]}\, C^{[\widetilde{41}]\,T'S'}_{[\tilde{4}]\,T''S''[1]}, \quad (19)$$

and $$C^{[\widetilde{42}]\,TS}_{[22]\,T''S''[\tilde{2}]\,T_2S_2} = \sum_{T'S'} U(T''\tfrac{1}{2}T\tfrac{1}{2}; T'T_2)\, U(S''\tfrac{1}{2}S\tfrac{1}{2}, S'S_2)\, C^{[\widetilde{42}]\,TS}_{[32]\,T'S'[1]}\, C^{[\widetilde{32}]\,T'S'}_{[22]\,T''S''[1]}. \quad (20)$$

For the functions (16B) and (16C) we make use of the fact that, with $r<s$ and

$$\mu = f_r - f_s + (s-r),$$

$$\left.\begin{aligned} P_{n-1,n}\Gamma(\tilde{r}\tilde{s}\tilde{r}_{n-2}\ldots\tilde{r}_1) &= -\frac{1}{\mu}\Gamma(\tilde{r}\tilde{s}\tilde{r}_{n-2}\ldots\tilde{r}_1) - \frac{\sqrt{(\mu^2-1)}}{\mu}\Gamma(\tilde{s}\tilde{r}\tilde{r}_{n-2}\ldots\tilde{r}_1),\\ P_{n-1,n}\Gamma(\tilde{s}\tilde{r}\tilde{r}_{n-2}\ldots\tilde{r}_1) &= -\frac{\sqrt{(\mu^2-1)}}{\mu}\Gamma(\tilde{r}\tilde{s}\tilde{r}_{n-2}\ldots\tilde{r}_1) + \frac{1}{\mu}\Gamma(\tilde{s}\tilde{r}\tilde{r}_{n-2}\ldots\tilde{r}_1).\end{aligned}\right\} \quad (21)$$

We may hence define the standard diagonalized adjoint Young-Yamanouchi representation by

$$\left.\begin{aligned} \Gamma(\{\tilde{r}\tilde{s}\}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) &= \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\Gamma(\tilde{r}\tilde{s}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) + \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\Gamma(\tilde{s}\tilde{r}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1),\\ \Gamma([\tilde{r}\tilde{s}]\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) &= \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\Gamma(\tilde{r}\tilde{s}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) - \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\Gamma(\tilde{s}\tilde{r}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1),\end{aligned}\right\} \quad (22)$$

such that $$\left.\begin{aligned} P_{n-1,n}\Gamma(\{\tilde{r}\tilde{s}\}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) &= -\Gamma(\{\tilde{r}\tilde{s}\}\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1),\\ P_{n-1,n}\Gamma([\tilde{r}\tilde{s}]\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1) &= +\Gamma([\tilde{r}\tilde{s}]\tilde{r}_{n-2}\ldots\tilde{r}_2\tilde{r}_1).\end{aligned}\right\} \quad (23)$$

It follows that we may express functions (16B) and (16C) as follows:

$$(\tilde{\text{B}}): \Gamma(\gamma^6[\widetilde{42}]\, TSM_T M_S \,|\, \tilde{2}\,\tilde{1}\,\tilde{r}_4\ldots\tilde{r}_1)$$
$$= \sqrt{\left(\frac{\mu-1}{2\mu}\right)} \sum_{T''S''T_2S_2} C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\, T''S''(\tilde{r}_4\ldots\tilde{r}_1), \gamma_5\gamma_6[\tilde{2}]\, T_2S_2, TSM_T M_S)$$
$$- \sqrt{\left(\frac{\mu+1}{2\mu}\right)} \sum_{T''S''T_2S_2} C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\widetilde{11}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\, T''S''(\tilde{r}_4\ldots\tilde{r}_1), \gamma_5\gamma_6[\widetilde{11}]\, T_2S_2, TSM_T M_S), \quad (18\text{B})$$

$$(\tilde{\mathrm{C}}):\ \Gamma(\gamma^6[\widetilde{42}]\,TSM_TM_S\,|\,\tilde{1}\,\tilde{2}\,\tilde{r}_4\ldots\tilde{r}_1)$$

$$=\sqrt{\left(\frac{\mu+1}{2\mu}\right)}\sum_{T''S''T_2S_2}C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}\,\Gamma(\gamma^4[\widetilde{31}]\,T''S''(\tilde{r}_4\ldots\tilde{r}_1),\gamma_5\gamma_6[\tilde{2}]\,T_2S_2,\,TSM_TM_S)$$

$$+\sqrt{\left(\frac{\mu-1}{2\mu}\right)}\sum_{T''S''T_2S_2}C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\widetilde{11}]\,T_2S_2}\,\Gamma(\gamma^4[\widetilde{31}]\,T''S''(\tilde{r}_4\ldots\tilde{r}_1),\gamma_5\gamma_6[\widetilde{11}]\,T_2S_2,TSM_TM_S),\tag{18C}$$

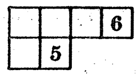
where, by comparing expansions, we have

$$C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}=\sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{T'S'}U(T''\tfrac{1}{2}T\tfrac{1}{2};\,T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]}\,C^{[\widetilde{41}]\,T'S}_{[\widetilde{31}]\,T''S''[1]}\tag{24a}$$

$$=\sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{T'S'}U(T''\tfrac{1}{2}T\tfrac{1}{2};\,T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{[\widetilde{42}]\,TS}_{[\widetilde{32}]\,T'S'[1]}\,C^{[\widetilde{32}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]},\tag{24b}$$

$$C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\widetilde{11}]\,T_2S_2}=-\sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{T'S'}U(T''\tfrac{1}{2}T\tfrac{1}{2};\,T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{[\widetilde{42}]\,TS}_{[\widetilde{41}]\,T'S'[1]}\,C^{[\widetilde{41}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]}\tag{25a}$$

$$=\sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{T'S'}U(T''\tfrac{1}{2}T\tfrac{1}{2};T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{[\widetilde{42}]\,TS}_{[\widetilde{32}]\,T'S'[1]}\,C^{[\widetilde{32}]\,T'S'}_{[\widetilde{31}]\,T''S''[1]}.\tag{25b}$$

Here, as before, μ is the axial distance from 6 to 5 in the scheme  and is equal to 3.

If we use $[f]=[f_1f_2f_3f_4]$ to describe the irreducible representations $\mathscr{H}_{[f]}$ of U_4, then, as stated in the Introduction, we may use $(g)=(g_1g_2g_3)=(f_1-f_2,f_2-f_3,f_3-f_4)$ to describe the corresponding representation $K_{(g)}$ of SU_4. The general result is then, introducing a symbol ρ to cover the possible occurrence of different states with the same TS in $K_{(g)}$,

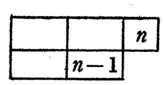
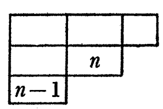
$$C^{(g)\,\rho\,TS}_{(g'')\,\rho''\,T''S''(010)\,T_2S_2}=\sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{T'S'\rho'}U(T''\tfrac{1}{2}T\tfrac{1}{2};T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{(g)\,\rho\,TS}_{(g'_a)\,\rho'\,T'S'[1]}\,C^{(g'_a)\,\rho'\,T'S'}_{(g'')\,\rho''\,T''S''[1]}\tag{26a}$$

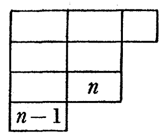
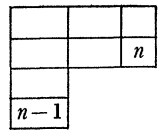
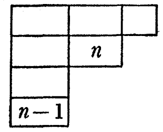
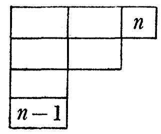
$$=\sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{T'S'\rho'}U(T''\tfrac{1}{2}T\tfrac{1}{2};T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{(g)\,\rho\,TS}_{(g'_b)\,\rho'\,T'S'[1]}\,C^{(g'_b)\,\rho'\,T'S'}_{(g'')\,\rho''\,T''S''[1]},\tag{26b}$$

$$C^{(g)\,\rho\,TS}_{(g'')\,\rho''\,T''S''(200)\,T_2S_2}=-\sqrt{\left(\frac{2\mu}{\mu+1}\right)}\sum_{T'S'\rho'}U(T''\tfrac{1}{2}T\tfrac{1}{2};\,T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{(g)\,\rho\,TS}_{(g'_a)\,\rho'\,T'S'[1]}\,C^{(g'_a)\,\rho'\,T'S'}_{(g'')\,\rho''\,T''S''[1]}\tag{27a}$$

$$=\sqrt{\left(\frac{2\mu}{\mu-1}\right)}\sum_{T'S'\rho'}U(T''\tfrac{1}{2}T\tfrac{1}{2};\,T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{(g)\,\rho\,TS}_{(g'_b)\,\rho'\,T'S'[1]}\,C^{(g'_b)\,\rho'\,T'S'}_{(g'')\,\rho''\,T''S''[1]},\tag{27b}$$

holding for the following sets of values of (g), (g''), (g'_a), (g'_b) with the appropriate value of μ indicated and corresponding, as shown in the last column, to the six possible positions of the numbers n and $n-1$ in the Young symmetry scheme such that they occur neither in the same column nor the same row:

(g)	(g'')	(g'_a)	(g'_b)	μ	
$(g_1g_2g_3)$	$(g_1,\,g_2-1,\,g_3)$	$(g_1-1,\,g_2,\,g_3)$	$(g_1+1,\,g_2-1,\,g_3)$	g_1+1	
	$(g_1+1,\,g_2,\,g_3-1)$	$(g_1+1,\,g_2-1,\,g_3)$	$(g_1,\,g_2+1,\,g_3-1)$	g_2+1	

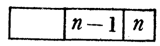
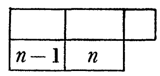
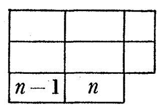
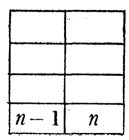
(g)	(g'')	(g'_a)	(g'_b)	μ	
	(g_1, g_2+1, g_3)	(g_1, g_2+1, g_3-1)	(g_1, g_2, g_3+1)	g_3+1	
	(g_1-1, g_2+1, g_3-1)	(g_1-1, g_2, g_3)	(g_1, g_2+1, g_3-1)	g_1+g_2+2	
	(g_1+1, g_2-1, g_3+1)	(g_1+1, g_2-1, g_3)	(g_1, g_2, g_3+1)	g_2+g_3+2	
	(g_1-1, g_2, g_3+1)	(g_1-1, g_2, g_3)	(g_1, g_2, g_3+1)	$g_1+g_2+g_3+3$	

In addition, the simple relation

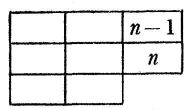
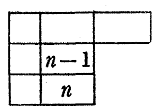
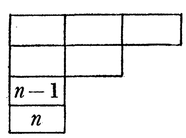
$$C^{(g)\rho TS}_{(g'')\rho''T''S''(G)\,T_2S_2} = \sum_{T'S'\rho'} U(T''\tfrac{1}{2}T\tfrac{1}{2};T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};S'S_2)\,C^{(g)\rho TS}_{(g')\rho'T'S'[1]}\,C^{(g')\rho'T'S'}_{(g'')\rho''T''S''[1]} \qquad (28)$$

holds for the following cases, corresponding to the partitions shown with n and $n-1$ either in the same row or the same column:

$(G) = (200)$

(g)	(g'')	(g')	
$(g_1g_2g_3)$	(g_1-2, g_2, g_3)	(g_1-1, g_2, g_3)	
	(g_1+2, g_2-2, g_3)	(g_1+1, g_2-1, g_3)	
	(g_1, g_2+2, g_3-2)	(g_1, g_2+1, g_3-1)	
	(g_1, g_2, g_3+2)	(g_1, g_2, g_3+1)	

$(G) = (010)$

(g)	(g'')	(g')	
$(0g_2g_3)$	$(0, g_2-1, g_3)$	$(1, g_2-1, g_3)$	
(g_10g_3)	$(g_1+1, 0, g_3-1)$	$(g_1, 1, g_3-1)$	
(g_1g_20)	$(g_1, g_2+1, 0)$	$(g_1, g_2, 1)$	

Coefficients sufficient to construct the wave functions for the whole of the nuclear p-shell satisfying these relations are given in tables 5*a* (n even) and 5*b* (n odd).

3. Construction of the totally antisymmetric wave function

We begin by describing in detail the construction of the totally antisymmetric states $\psi(p^6[42]\ TS\kappa LM_TM_SM_L \mid 654321)$ for the p^6 configuration. If $n_{[f]}$ denotes the dimension of the irreducible representation $R_{[f]}$ of the symmetric group S_6 of permutations of the six particles, we have

$$\psi(p^6[42]\ TS\kappa LM_TM_SM_L \mid 654321)$$

$$= \sqrt{\left(\frac{1}{n_{[42]}}\right)} \sum_{(r)} \Phi(p^6[42]\ \kappa LM_L \mid r_6 r_5 \ldots r_1)\ \Gamma(\gamma^6[\widetilde{42}]\ TSM_TM_S \mid \tilde{r}_6\tilde{r}_5 \ldots \tilde{r}_1)$$

$$= \sqrt{\left(\frac{1}{n_{[42]}}\right)} \sum_{r_6 r_5} \sum_{(r'')} \Phi(p^6[42]\ \kappa LM_L \mid r_6 r_5 r'')\ \Gamma(\gamma^6[\widetilde{42}]\ TSM_TM_S \mid \tilde{r}_6\tilde{r}_5 r'')$$

$$= \sqrt{\left(\frac{1}{n_{[42]}}\right)} \sum_{\substack{r''L''T''S''\\ T_2S_2L_2}} \Big[C^{[42]\,\kappa L}_{[4]\,L''[2]\,L_2}\, C^{[\widetilde{42}]\,TS}_{[\tilde{4}]\,T''S''[\tilde{2}]\,T_2S_2}$$

$$\times \Phi(p^4[4]\ L''(r''), p_5p_6[2]\ L_2, LM_L)\ \Gamma(\gamma^4[\tilde{4}]\ T''S''(r''), \gamma_5\gamma_6[\tilde{2}]\ T_2S_2, TSM_TM_S)$$

$$+ C^{[42]\,\kappa L}_{[22]\,L''[2]\,L_2}\, C^{[\widetilde{42}]\,TS}_{[22]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(p^4[22]\ L''(r''), p_5p_6[2]\ L_2, LM_L)$$

$$\times \Gamma(\gamma^4[\widetilde{22}]\ T''S''(r''), \gamma_5\gamma_6[\tilde{2}]\ T_2S_2, TSM_TM_S)$$

$$+ \Big\{ \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[2]\,L_2}\, \Phi(p^4[31]\ L''(r''), p_5p_6[2]\ L_2, LM_L)$$

$$- \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[11]\,L_2}\, \Phi(p^4[31]\ L''(r''), p_5p_6[11]\ L_2, LM_L) \Big\}$$

$$\times \Big\{ \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\, C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\ T''S''(r''), \gamma_5\gamma_6[\tilde{2}]\ T_2S_2, TSM_TM_S)$$

$$- \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\, C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\widetilde{11}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\ T''S''(r''), \gamma_5\gamma_6[\widetilde{11}]\ T_2S_2, TSM_TM_S) \Big\}$$

$$+ \Big\{ \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[2]\,L_2}\, \Phi(p^4[31]\ L''(r''), p_5p_6[2]\ L_2, LM_L)$$

$$+ \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[11]\,L_2}\, \Phi(p^4[31]\ L''(r''), p_5p_6[11]\ L_2, LM_L) \Big\}$$

$$\times \Big\{ \sqrt{\left(\frac{\mu+1}{2\mu}\right)}\, C^{[\widetilde{42}]\,TS}_{[\widetilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\ T''S''(r''), \gamma_5\gamma_6[\tilde{2}]\ T_2S_2, TSM_TM_S)$$

$$+ \sqrt{\left(\frac{\mu-1}{2\mu}\right)}\, C^{[\widetilde{42}]\,TS}_{[31]\,T''S''[\widetilde{11}]\,T_2S_2}\, \Gamma(\gamma^4[\widetilde{31}]\ T''S''(r''), \gamma_5\gamma_6[\widetilde{11}]\ T_2S_2, TSM_TM_S) \Big\} \Big]. \tag{29}$$

We see that the transformation coefficients diagonalizing the Young-Yamanouchi representation disappear in the final form due to orthonormality, and carrying out the summation over (r'') to obtain antisymmetric states in four particles we have

$$\Psi'(p^6[42]\,TS\kappa LM_T M_S M_L\,|\,654321)$$
$$= \sum_{\substack{T''S''L''\\ T_2S_2L_2}} \Bigg[\sqrt{\left(\frac{n_{[4]}}{n_{[42]}}\right)}\, C^{[42]\,\kappa L}_{[4]\,L''[2]\,L_2}\, C^{[\tilde{42}]\,TS}_{[4]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(p^4[4]\,T''S''L'', p^2[2]\,T_2S_2L_2, TSLM_TM_SM_L)$$
$$+\sqrt{\left(\frac{n_{[22]}}{n_{[42]}}\right)}\, C^{[42]\,\kappa L}_{[22]\,L''[2]\,L_2}\, C^{[\tilde{42}]\,TS}_{[\tilde{22}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(p^4[22]\,T''S''L'', p^2[2]\,T_2S_2L_2, TSLM_TM_SM_L)$$
$$+\sqrt{\left(\frac{n_{[31]}}{n_{[42]}}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[2]\,L_2}\, C^{[\tilde{42}]\,TS}_{[\tilde{31}]\,T''S''[\tilde{2}]\,T_2S_2}\, \Phi(p^4[31]\,T''S''L'', p^2[2]\,T_2S_2L_2, TSLM_TM_SM_L)$$
$$+\sqrt{\left(\frac{n_{[31]}}{n_{[42]}}\right)}\, C^{[42]\,\kappa L}_{[31]\,L''[11]\,L_2}\, C^{[\tilde{42}]\,TS}_{[\tilde{31}]\,T''S''[\tilde{11}]\,T_2S_2}\, \Phi(p^4[31]\,T''S''L'', p^2[11]\,T_2S_2L_2, TSLM_TM_SM_L)\Bigg]. \quad (30)$$

The general result for n nucleons in the same l shell is clearly

$$\Psi'(l^n[f]\,\rho TS\kappa LM_TM_SM_L\,|\,n, n-1, 2, \ldots, 1)$$
$$= \sum_{\substack{f''\rho''T''S''\kappa''L''\\ T_2S_2L_2}} \sqrt{\left(\frac{n_{[f'']}}{n_{[f]}}\right)}\, C^{[f]\,\kappa L}_{[f'']\,\kappa''L''[f_2]\,L_2}\, C^{[\tilde{f}]\,\rho TS}_{[\tilde{f}'']\,\rho''T''S''[\tilde{f}_2]\,T_2S_2}$$
$$\times \Phi(l^{n-2}[f'']\,\rho''T''S''\kappa''L'', l^2[f_2]\,T_2S_2L_2, TSLM_TM_SM_L). \quad (31)$$

(f_2 need not appear in the summation, since for the l^2 configuration f_2 is determined by T_2, S_2 and L_2.) Thus, as with the $\langle n\,|\,n-1, 1\rangle$ coefficients, the total $\langle n\,|\,n-2, 2\rangle$ fractional parentage coefficients are direct products of weight factors (equal to the square root of the ratios of dimensions of representations of permutation groups), orbital factors and charge-spin factors. These three factors are listed separately in tables 3, 4, 5 for the nuclear p-shell, use being made of the special unitary group representation symbols to simplify the tabulation.

It may be verified that the orbital and charge-spin coefficients here tabulated satisfy the generalized Racah reciprocal relation (see Jahn in Addendum to Jahn & van Wieringen (1951) for a special case, the general proof will be published elsewhere). These relations may be written in the new notation as follows:

$$C^{(g_1g_2)\,\kappa L}_{(g_1''g_2'')\,\kappa''L''(G_1G_2)L_2} = (-1)^{L+L''+L_2+x}\sqrt{\frac{N_g}{N_{g''}}}\sqrt{\frac{2L''+1}{2L+1}}\, C^{(g_2''g_1'')\,\kappa''L''}_{(g_2g_1)\,\kappa L(G_1G_2)\,L_2}, \quad (32)$$

$$C^{(g_1g_2g_3)\,\rho TS}_{(g_1''g_2''g_3'')\,\rho''T''S''(G_1G_2G_3)\,T_2S_2} = (-1)^{T+S+T''+S''+T_2+S_2+x}\sqrt{\left\{\frac{N_g}{N_{g''}}\,\frac{(2T''+1)\,(2S''+1)}{(2T+1)\,(2S+1)}\right\}}\, C^{(g_3''g_2''g_1'')\,\rho''T''S''}_{(g_3g_2g_1)\,\rho TS(G_1G_2G_3)\,T_2S_2}, \quad (33)$$

where N_g is used to denote the dimension of the representation $\mathscr{K}_{(g)}$ of the special unitary group in question. These relations hold in this simple form only if $K(g)$ occurs with unit multiplicity in the reduced form of the product representation $K_{(g'')}\times K_{(G)}$. The (amended) phases of this and the previous paper (part A) are such that x is always zero in the reciprocal relations, when due account is taken of the additional change of sign occurring with starred states (see Addendum to part A for a discussion of starred states).

For higher shells ($l \geqslant 2$) it follows from Racah's work (1949) that the orbital coefficients $C^{[f]\kappa L}_{[f'']\kappa'' L''[f_2] L_2}$ may be further factorized by consideration of the orthogonal group R_{2l+1} (see Jahn (1951) for tabulated values of the $\langle d^4 \mid d^2 d^2 \rangle$ coefficients exhibiting this factorization, some of the $\langle d^5 \mid d^3 d^2 \rangle$ coefficients have been evaluated by J. Hope (unpublished)). In the special case $l = 3$ Racah (1949) has shown that a still further factorization is possible, each such factorization leading to a partial specification of the observables κ distinguishing states of the same L belonging to the same representation $\mathscr{H}_f$ of U_{2l+1}. Similar considerations apply to nuclear or atomic wave functions in jj-coupling, the additional factorization being made by means of the unitary symplectic group (cf. Flowers 1952) and tables of $\langle j^n \mid j^{n-2} j^2 \rangle$ spin-orbital coefficients with $j \leqslant \frac{3}{2}$ have been prepared (Jahn, unpublished), such that the corresponding representations of the unitary and symplectic groups are in a standard specified form. All these coefficients satisfy the generalization of the Racah reciprocal relation.

The method of $\langle n \mid n-2, 2 \rangle$ fractional parentage coefficients is applicable also to mixed configurations. We have, with $\sum_{i=1}^{r} n_i = n$,

$$
\begin{aligned}
&\Psi(l_1^{n_1} l_2^{n_2} \ldots l_r^{n_r}[f]\,\rho TS\kappa LM_T M_S M_L \mid n, n-1, \ldots, 2, 1) \\
&\quad = \sum_{\substack{i=1 \\ n_i \geqslant 2}}^{r} \sum_{\substack{f''\rho'' T'' S''\kappa'' L'' \\ T_2 S_2 L_2}} \sqrt{\left(\frac{n_{f''}}{n_f}\right)}\, C^{l_1^{n_1} \ldots l_r^{n_r}[f]\,\kappa L}_{l_1^{n_1} \ldots l_i^{n_i-2} \ldots l_r^{n_r}[f'']\,\kappa'' L'' l_i^2 [f_2]\, L_2}\, C^{[\tilde{f}]\,\rho TS}_{[\tilde{f}'']\,\rho'' T'' S''[\tilde{f}_2]\, T_2 S_2} \\
&\qquad \times \Phi(l_1^{n_1} \ldots l_i^{n_i-2} \ldots l_r^{n_r}[f'']\,\rho'' T'' S''\kappa'' L'', l_i^2[f_2]\, T_2 S_2 L_2, TSLM_T M_S M_L) \\
&\qquad + \sum_{i<j}^{r} \sum_{\substack{f''\rho'' T'' S''\kappa'' L'' \\ f_2 T_2 S_2 L_2}} \sqrt{\left(\frac{n_{f''}}{n_f}\right)}\, C^{l_1^{n_1} \ldots l_r^{n_r}[f]\,\kappa L}_{l_1^{n_1} \ldots l_i^{n_i-1} \ldots l_j^{n_j-1} \ldots l_r^{n_r}[f'']\,\kappa'' L'' l_i l_j [f_2]\, L_2}\, C^{[\tilde{f}]\,\rho TS}_{[\tilde{f}'']\,\rho'' T'' S''[\tilde{f}_2]\, T_2 S_2} \\
&\qquad \times \Phi(l_1^{n_1} \ldots l_i^{n_i-1} \ldots l_j^{n_j-1} \ldots l_r^{n_r}[f'']\,\rho'' T'' S''\kappa'' L'', l_i l_j [f_2]\, T_2 S_2 L_2, TSLM_T M_S M_L). \qquad (34)
\end{aligned}
$$

(f_2 must now be included in the last summation, since with inequivalent particles the same L_2 occurs with both symmetrical ($[f_2] = [2]$) and antisymmetrical ($[f_2] = [11]$) two-particle orbital states. In (34), for definiteness, the two-particle totally antisymmetric states arising from l_i^2 and $l_i l_j$ may be taken to be occupied by particles n and $n-1$.) Expressions for these more general $\langle n \mid n-2, 2 \rangle$ fractional parentage coefficients have been given by J. P. Elliott in his thesis, for the case of a configuration consisting of one incomplete shell and a number of closed shells.

Elliott has shown how, given such a set of $\langle n \mid n-2, 2 \rangle$ fractional parentage coefficients, the matrix elements of any two-body interaction, central or non-central, may be evaluated. In the following paragraph we show merely, following Hope (1950), how the $\langle n \mid n-2, 2 \rangle$ coefficients for the nuclear p-shell, here tabulated, may be used in a derivation, alternative to that of Hund (1937), Feenberg & Wigner (1937), Feenberg & Phillips (1937), Racah (1941, 1950), of the known central force matrix for the nuclear p-shell. For the manner in which these coefficients have been used to evaluate the matrix elements of non-central forces in p-shell nuclei see papers by J. P. Elliott (1953) (^{7}Li, ^{10}B) and W. J. Robinson (1953) (^{14}N). The charge-spin factors listed in this paper may be used unchanged in problems involving interconfigurational mixing, and they are being used in this way, by other members of the Southampton group, in total binding energy calculations of light nuclei with central and tensor forces.

4. The p-shell charge-symmetric central force matrix

The complete energy matrix for a charge-symmetric two-body central force interaction in the nuclear p-shell has been given by Racah (1950), completing earlier work of Hund (1937), Feenberg & Wigner (1937), Feenberg & Phillips (1937) and Racah (1942). The complete matrix had also been obtained by the senior author and H. van Wieringen (unpublished). In terms of the coefficients a_0, a_σ, a_τ, $a_{\sigma\tau}$ of the general charge-symmetric two-body interaction

$$H_{12} = J(r_{12})\,(a_0 + a_\sigma(\vec{\sigma}_1\vec{\sigma}_2) + a_\tau(\vec{\tau}_1\vec{\tau}_2) + a_{\sigma\tau}(\vec{\sigma}_1\vec{\sigma}_2)\,(\vec{\tau}_1\vec{\tau}_2) \tag{35}$$

($J(r_{12})$ could be different for each type of interaction, requiring different A's and B's below), and with the notation of Hund (1937) for the two radial integrals

$$A = F_0 + 4F_2, \quad B = 3F_2 \tag{36}$$

(see Swiatecki (1951), or Elliott's thesis, for expressions for these), A and B being respectively identical with the L and K of Feenberg & Phillips (1937), the Racah expression may be written as

$$\begin{aligned}
&\langle p^n[f]\,\rho TS\kappa LM_T M_S M_L \,|\, \sum_{i<j} H_{ij} \,|\, p^n[f']\,\rho' TS\kappa' LM_T M_S M_L\rangle \\
&= \delta_{ff'}\delta_{\rho\rho'}\delta_{\kappa\kappa'}\Big[A\Big\{\frac{n(n-1)}{2}\,a_0 + \Big(-\frac{3n}{2}+2S(S+1)\Big)\,a_\sigma + \Big(-\frac{3n}{2}+2T(T+1)\Big)\,a_\tau \\
&\quad + \Big(-\frac{n(n-7)}{2} - 4(\alpha-\beta) - 2T(T+1) - 2S(S+1)\Big)\,a_{\sigma\tau}\Big\} \\
&\quad + B\Big\{(-n(n-2) + 2(\alpha-\beta) - \tfrac{1}{2}L(L+1))\,a_0 \\
&\quad + \Big(-\frac{n(n-8)}{2} - 2(\alpha-\beta) + \tfrac{1}{2}L(L+1) - \tfrac{5}{2}T(T+1) - \tfrac{7}{2}S(S+1)\Big)\,a_\sigma \\
&\quad + \Big(-\frac{n(n-8)}{2} - 2(\alpha-\beta) + \tfrac{1}{2}L(L+1) - \tfrac{7}{2}T(T+1) - \tfrac{5}{2}S(S+1)\Big)\,a_\tau \\
&\quad + (-12n + 6(\alpha-\beta) + \tfrac{3}{2}L(L+1) + 6T(T+1) + 6S(S+1))\,a_{\sigma\tau}\Big\}\Big] \\
&\quad + 2B(a_\sigma - a_\tau)\,\langle p^n[f]\,\rho TS\kappa LM_T M_S M_L \,|\, X \,|\, p^n[f']\,\rho' TS\kappa' LM_T M_S M_L\rangle.
\end{aligned} \tag{37}$$

Here α, β are the symmetry symbols of Hund (1937) (α is the number of different pairs of squares occurring in the same row of the Young tableau describing the permutation symmetry of the orbital state, β is the number of different pairs of squares occurring in the same column of this tableau), and

$$X = \sum_{i<j} X_{ij} \tag{38}$$

is a two-particle charge-symmetric central interaction which is diagonal for the two-particle states with the following values:

	^{13}S	^{31}S	^{13}D	^{31}D	^{11}P	^{33}P	
$X_{12} =$	$\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	(39)

By comparing eigenvalues, one may verify easily that in terms of tensor operators

$$X_{ij} = \tfrac{15}{16}\{(u_i^{(1)} \cdot u_j^{(1)}) - (u_i^{(2)} \cdot u_j^{(2)})\}\{(\vec{\tau}_i \cdot \vec{\tau}_j) - (\vec{\sigma}_i \cdot \vec{\sigma}_j)\}. \tag{40}$$

Here

$$(u_i^{(r)} \cdot u_j^{(r)}) = \sum_q (-1)^q\, u_q^{(r)}(i)\, u_{-q}^{(r)}(j) \tag{41}$$

is the scalar product of the purely orbital unit tensor operators $u_q^{(r)}(i)$ and $u_q^{(r)}(j)$, introduced by Racah (1942), which act on the orbital states $\phi_m^l(i)$ and $\phi_m^l(j)$ of particles i and j according to

$$u_q^{(r)}(i)\, \phi_m^l(i) = \frac{1}{\sqrt{(2l+1)}}\, C_{lmrq}^{lm+q}\, \phi_{m+q}^l(i). \tag{42}$$

$\vec{\sigma}_i$ is the usual Pauli spin vector operator for particle i, $\vec{\tau}_i$ the corresponding symbolic isotopic spin vector operator.

It may be shown in the manner of Racah (1951, Princeton lectures) that the two-particle interaction X is of symmetry type (22) (020) ^{11}S, where the first bracket is the symbol $(g_1g_2) = (f_1-f_2, f_2-f_3)$ of an irreducible representation of the group SU_3 of special unitary transformations of the single-particle orbital states and the second bracket is the symbol $(g_1g_2g_3) = (f_1-f_2, f_2-f_3, f_3-f_4)$ of an irreducible representation of the group SU_4 of special unitary transformations in the space of the combined charge and spin states of the single particle. From the following reductions of the product representations

$$(22)\times(g_1g_2) \quad \text{and} \quad (020)\times(g_1g_2g_3),$$

where (g_1g_2), $(g_1g_2g_3)$ take on the values occurring for the states of the nuclear p^n configuration, selection rules may be deduced for the matrix elements of X, and these show that only those elements tabulated in table 6 below can be different from zero.

Reduction of the product representation $(22)\times(g_1g_2)$ *for the group* SU_3

$$(22)\times(00) = (22),$$

$$(22)\times(10) = (32)+(13)+(21),$$

$$(22)\times(20) = (42)+(23)+(31)+(04)+(12)+(20),$$

$$(22)\times(11) = (33)+(41)+(14)+(22)+(30)+(03)+(11),$$

$$(22)\times(30) = (52)+(33)+(41)+(14)+(22)+(30)+(03)+(11),$$

$$(22)\times(21) = (43)+(51)+(24)+2(32)+(40)+2(13)+2(21)+(02)+(10)+(05),$$

$$(22)\times(40) = (62)+(43)+(51)+(24)+(32)+(40)+(13)+(21)+(02),$$

$$(22)\times(31) = (53)+(61)+(34)+2(42)+(50)+(15) +2(23)+2(31)+(04)+2(12)+(20)+(01),$$

$$(22)\times(22) = (44)+(52)+(25)+(60)+(06)+2(33)+2(41) +2(14)+(03)+(30)+(11)+(00)+3(22).$$

Reduction of the product representation $(020)\times(g_1g_2g_3)$ *for the group* SU_4

$$(020)\times(000) = (020),$$

$$(020)\times(200) = (220)+(111)+(002),$$

$$(020)\times(010) = (030)+(111)+(010),$$

$$(020)\times(100) = (120)+(011),$$

$$(020)\times(110) = (130)+(210)+(102)+(110)+(001)+(021),$$

$$(020)\times(101) = (121)+(210)+(012)+(020)+(101),$$

$$(020)\times(020) = (040)+(121)+(202)+(020)+(000)+(101),$$

$$(020)\times(300) = (320)+(210)+(102),$$

$$(020)\times(210) = (230)+(311)+(202)+(210)+(101)+(012)+(121),$$

$$(020)\times(201) = (221)+(310)+(112)+(120)+(201)+(003)+(011),$$

$$(020)\times(120) = (140)+(221)+(302)+(031)+(112)+(120)+(201)+(011)+(100),$$

$$(020)\times(111) = (131)+(212)+(220)+(022)+(301) \\ +(103)+(030)+2(111)+(200)+(002)+(010),$$

$$(020)\times(030) = (050)+(131)+(212)+(111)+(010)+(030).$$

[These reductions may be obtained by considering the corresponding representations of the unitary groups and then using the theorem of Littlewood (1940, theorem V, p. 94) as used previously (Jahn 1950). Since the contragredient representation is obtained by reversing the order of the g symbols (see above), and since the representations of the interaction are self-contragredient, it follows that if

$$(22)\times(g_1g_2) = \Sigma a_{kl}(g_kg_l), \tag{43}$$

then

$$(22)\times(g_2g_1) = \Sigma a_{kl}(g_lg_k), \tag{44}$$

and so also from

$$(020)\times(g_1g_2g_3) = \Sigma a_{klm}(g_kg_lg_m), \tag{45}$$

follows

$$(020)\times(g_3g_2g_1) = \Sigma a_{klm}(g_mg_lg_k). \tag{46}$$

Thus we need only list the reductions for those representations (g_1g_2) with $g_1\geqslant g_2$ or those $(g_1g_2g_3)$ with $g_1\geqslant g_3$.]

Although the matrix X has already been given by Racah, he did not specify the phases of his wave functions, so that the signs of the non-diagonal elements are in general different from those appropriate to the standard (amended) phases of Jahn & van Wieringen. We have recalculated this matrix using the $\langle n\,|\,n-2,2\rangle$ fractional parentage coefficients, using formulae given below, and the values are given in table 6, which differs only in the overall sign of certain elements from Racah's table. As Racah points out, the elements can be factorized into a purely orbital and purely charge-spin component, and it is possible to choose these components so that they depend only on the representation symbols of the appropriate special unitary groups; they are further invariant with respect to a change to the contragredient representation with the possible exception of a change of sign when one starred state of a self-contragredient representation is involved. In going from corresponding states of the p^n configuration to those of the p^{12-n} configuration, which belong to contragredient representations, there is, in fact, with the Jahn & van Wieringen standard (amended) phases only one matrix element which changes sign; this is the element

$$\begin{aligned}\langle p^4[211]\,(210)\,(10)\,{}^{33}P \,|\, X \,|\, p^4[31]\,(101)\,(21)\,{}^{\overset{*}{33}}P\rangle \\ = -\langle p^8[332]\,(012)\,(01)\,{}^{33}P \,|\, X \,|\, p^8[431]\,(101)\,(12)\,{}^{\overset{*}{33}}P\rangle \\ = 2. \end{aligned} \tag{47}$$

All other matrix elements are the same for corresponding states of the p^n and p^{12-n} configurations.

The matrix elements of X were derived in the following manner (following Hope 1950). We may write the wave functions for the nuclear l^n configuration in the form

$$\begin{aligned}\Psi(l^n[f]\,\rho TS\kappa LM_T M_S M_L) = {} & \sum_{f''_s} \sqrt{\frac{n_{f''_s}}{n_f}} \sum_{\rho''T''S''\kappa''L''} \sum_{T_2 S_2 L_2} C^{f\kappa L}_{f''_s\kappa''L''[2]\,L_2}\, C^{\tilde{f}\rho TS}_{\tilde{f}''_s\rho''T''S''[\tilde{2}]\,T_2S_2} \\ & \times \Phi(l^{n-2}[f''_s]\,\rho''T''S''\kappa''L'', l^2[2]\,T_2S_2L_2, TSLM_TM_SM_L) \\ & + \sum_{f''_a} \sqrt{\frac{n_{f''_a}}{n_f}} \sum_{\rho''T''S''\kappa''L''} \sum_{T_2 S_2 L_2} C^{f\kappa L}_{f''_a\kappa''L''[11]\,L_2}\, C^{\tilde{f}\rho TS}_{\tilde{f}''_a\rho''T''S''[\tilde{1}\tilde{1}]\,T_2S_2} \\ & \times \Phi(l^{n-2}[f''_a]\,\rho''T''S''\kappa''L'', l^2[11]\,T_2S_2L_2, TSLM_TM_SM_L), \end{aligned} \tag{48}$$

where f''_s runs through those partitions of $n-2$ which occur with the partition [2], whilst f''_a runs through the partitions of $n-2$ which occur with the partition [11]. $n_f, n_{f''_s}, n_{f''_a}$ are the respective dimensions of the representations $R_{[f]}$ of S_n, $R_{[f''_s]}$, $R_{[f''_a]}$ of S_{n-2}. We have then

$$\sum_{f''_s} n_{f''_s} + \sum_{f''_a} n_{f''_a} = n_f, \tag{49}$$

$$\sum_{f''_s} n_{f''_s} - \sum_{f''_a} n_{f''_a} = \chi_{12} = \frac{2n_f}{n(n-1)}\,(\alpha-\beta), \tag{50}$$

where χ_{12} is the character for a transposition in the representation R_f of S_n, and $\alpha-\beta$ is the difference of the Hund symmetry symbols for this representation. Since the interaction $X_{n-1,\,n}$ has non-vanishing matrix elements only for the states of symmetry type [2] of particles n and $n-1$, we have

$$\begin{aligned}\langle p^n[f]\,\rho TS\kappa LM_TM_SM_L \,|\, X_{n-1,\,n} \,|\, p^n[f']\,\rho' TS\kappa' LM_TM_SM_L\rangle \\ = \sum_{f''_s} \frac{n_{f''_s}}{\sqrt{(n_f n_{f'})}} \sum_{\rho''T''S''\kappa''L''} \sum_{T_2S_2L_2} C^{f\kappa L}_{f''_s\kappa''L''[2]\,L_2}\, C^{f'\kappa' L}_{f''_s\kappa''L''[2]\,L_2} \\ \times C^{\tilde{f}\rho TS}_{\tilde{f}''_s\rho''T''S''[\tilde{2}]\,T_2S_2}\, C^{\tilde{f}'\rho' TS}_{\tilde{f}''_s\rho''T''S''[\tilde{2}]\,T_2S_2} \langle p^2[2]\,T_2S_2L_2 \,|\, X_{n-1,\,n} \,|\, p^2[2]\,T_2S_2L_2\rangle. \end{aligned} \tag{51}$$

Since the wave functions are totally antisymmetric we may obtain the matrix of the total interaction X by multiplying by $\frac{1}{2}[n(n-1)]$ and, inserting the numerical values of the two-particle matrix elements from (39), we have

$$\langle p^n[f]\,\rho TS\kappa LM_T M_S M_L\,|\,X\,|\,p^n[f']\,\rho' TS\kappa' LM_T M_S M_L\rangle$$

$$=\frac{n(n-1)}{2}\sum_{f''_s}\frac{n_{f''_s}}{\sqrt{(n_f n_{f'})}}\Big[\sum_{\rho''T''S''}\{C^{\tilde{f}\rho TS}_{f''_s\rho''T''S''[\tilde{2}]01}C^{\tilde{f}'\rho' TS}_{f''_s\rho''T''S''[\tilde{2}]01}-C^{\tilde{f}\rho TS}_{f''_s\rho''T''S''[\tilde{2}]10}C^{\tilde{f}'\rho' TS}_{f''_s\rho''T''S''[\tilde{2}]10}\}$$

$$\times\sum_{\kappa''L''}\{\tfrac{5}{2}C^{f\kappa L}_{f''_s\kappa''L''[2]S}C^{f'\kappa' L'}_{f''_s\kappa''L''[2]S}-\tfrac{1}{2}C^{f\kappa L}_{f''_s\kappa''L''[2]D}C^{f'\kappa' L}_{f''_s\kappa''L''[2]D}\}\Big]. \quad (52)$$

We now make use of relations included in the following list of relations satisfied by the $\langle n\,|\,n-2, 2\rangle$ parentage coefficients.

Orthonormality of the orbital coefficients:

$$\sum_{\kappa''L''L_2}C^{f\kappa L}_{f''\kappa''L''f_2L_2}C^{f'\kappa' L}_{f''\kappa''L''f_2L_2}=\delta(ff')\,\delta(\kappa\kappa'). \quad (53)$$

Orthonormality of the charge-spin coefficients:

$$\sum_{\rho''T''S''T_2S_2}C^{f\rho TS}_{f''\rho''T''S''f_2T_2S_2}C^{f'\rho' TS}_{f''\rho''T''S''f_2T_2S_2}=\delta(ff')\,\delta(\rho\rho'). \quad (54)$$

Charge-spin relations, following from the relations

$$\sum_{i<j}P^\sigma_{ij}=\frac{n(n-4)}{4}+S(S+1),\quad \sum_{i<j}P^\tau_{ij}=\frac{n(n-4)}{4}+T(T+1), \quad (55)$$

$$\frac{n(n-1)}{2n_f}\sum_{f''_s}n_{f''_s}\sum_{\rho''T''S''}\{C^{f\rho TS}_{f''_s\rho''T''S''[\tilde{2}]01}C^{f\rho' TS}_{f''_s\rho''T''S''[\tilde{2}]01}-C^{f\rho TS}_{f''_s\rho''T''S''[\tilde{2}]10}C^{f\rho' TS}_{f''_s\rho''T''S''[\tilde{2}]10}\}$$

$$=\delta(\rho\rho')\,\tfrac{1}{2}\{S(S+1)-T(T+1)\}, \quad (56)$$

$$\frac{n(n-1)}{2n_f}\sum_{f''_a}n_{f''_a}\sum_{\rho''T''S''}\{C^{f\rho TS}_{f''_a\rho''T''S''[\tilde{1}\tilde{1}]00}C^{f\rho' TS}_{f''_a\rho''T''S''[\tilde{1}\tilde{1}]00}-C^{f\rho TS}_{f''_a\rho''T''S''[\tilde{1}\tilde{1}]11}C^{f\rho' TS}_{f''_a\rho''T''S''[\tilde{1}\tilde{1}]11}\}$$

$$=-\delta(\rho\rho')\Big\{\frac{n(n-4)}{4}+\frac{T(T+1)}{2}+\frac{S(S+1)}{2}\Big\}. \quad (57)$$

Orbital relations, following from

$$\frac{1}{\hbar^2}\sum_{i<j}(\vec{l}_i\,.\,\vec{l}_j)=\frac{L(L+1)}{2}-n\frac{l(l+1)}{2}, \quad (58)$$

(together with use of (49) and (50))

$$\frac{n(n-1)}{2n_f}\sum_{f''_s}n_{f''_s}\sum_{\kappa''L''}C^{f\kappa L}_{f''_s\kappa''L''[2]S}C^{f\kappa' L}_{f''_s\kappa''L''[2]S}=\delta(\kappa\kappa')\Big[\frac{n}{3}+\frac{\alpha-\beta}{3}-\frac{L(L+1)}{6}\Big], \quad (59)$$

$$\frac{n(n-1)}{2n_f}\sum_{f''_s}n_{f''_s}\sum_{\kappa''L''}C^{f\kappa L}_{f''_s\kappa''L''[2]D}C^{f\kappa' L}_{f''_s\kappa''L''[2]D}=\delta(\kappa\kappa')\Big[\frac{n(3n-7)}{12}+\frac{\alpha-\beta}{6}+\frac{L(L+1)}{6}\Big]. \quad (60)$$

We may then transform (52) into the form

$$\langle p^n[f]\,\rho TS\kappa LM_T M_S M_L\,|\,X\,|\,p^n[f']\,\rho' TS\kappa' LM_T M_S M_L\rangle$$

$$=\tfrac{1}{4}\{T(T+1)-S(S+1)\}\,\delta(ff')\,\delta(\kappa\kappa')\,\delta(\rho\rho')+\tfrac{3}{2}n(n-1)\sum_{f''_s}\frac{n_{f''_s}}{\sqrt{(n_f n_{f'})}}\sum_{\kappa''L''}C^{f\kappa L}_{f''_s\kappa''L''[2]S}C^{f'\kappa' L}_{f''_s\kappa''L''[2]S}$$

$$\times\sum_{\rho''T''S''}\{C^{\tilde{f}\rho TS}_{\tilde{f}''_s\rho''T''S''[\tilde{2}]01}C^{\tilde{f}'\rho' TS}_{\tilde{f}''_s\rho''T''S''[\tilde{2}]01}-C^{\tilde{f}\rho TS}_{f''_s\rho''T''S''[\tilde{2}]10}C^{\tilde{f}'\rho' TS}_{f''_s\rho''T''S''[\tilde{2}]10}\}. \quad (61)$$

This was the final form used in the evaluation. It does not exhibit directly the factorization into the simple product of a purely orbital and purely charge-spin part, although this can be verified in each case. κ is used to distinguish states of the same L occurring with given $[f]$ (in the p-shell this only occurs for D_{I} and D_{II} of [42] which have been so chosen (see part A) that D_{II} is a starred state and D_{I} is unstarred) and ρ distinguishes states of the same TS occurring with a given $[\tilde{f}]$ (in the p-shell the only case occurs with $[\widetilde{321}]$ $^{33}\Gamma_{\mathrm{I}}$ and $^{33}\Gamma_{\mathrm{II}}$ where, again, one ($^{33}\Gamma_{\mathrm{II}}$) is starred and the other unstarred). As mentioned in part A, the states so defined are such that X is diagonal both with respect to κ and to ρ, i.e. the central force couplings between [42] TSD_{I} and [42] TSD_{II} and between [321] $^{33\mathrm{I}}L$ and [321] $^{33\mathrm{II}}L$ are zero.

Conclusions

The present position of the theory of nuclear structure has been admirably summarized recently by E. P. Wigner in his Brazil address (1952). The problem is one of calculating in detail the consequences of a two-body central and tensor force, and involves, in the shell model, a considerable amount of interconfigurational mixing. The tables presented here for one configuration represent one tool in this large programme. An important extension required for the light nuclei is the calculation of the $\langle n \mid n-2, 2\rangle$ parentage coefficients for the $s^m p^n$ configurations. General expressions for these in the special case of $s^4 p^n$ have been given already by Elliott in his thesis; other coefficients have been calculated and further calculations are being carried out at Southampton. A further extension required relates to the group-theoretical classification of the types of tensor force matrix elements in these mixed configurations. No calculation of the relative positions of the levels can be useful unless it is shown that the wave functions and interaction used give a reasonable total binding energy. Work is in progress on ^{6}Li, ^{7}Li, ^{9}Be and ^{14}N with such mixed configurations, extending previous work of J. P. Elliott and W. J. Robinson for one configuration, using the Pease & Feshbach (1951) mixture of central and tensor forces (which has been shown to give good binding energies for the deuteron, triton and α-particle).† Trial calculations with mixed s^4 and s^2p^2 single-particle configurations for ^{4}He carried out by P. G. Wakely (1953) suggest that good quantitative agreement with experiment can be achieved in this way. Compared with the *ad hoc* assumptions of the jj-coupling model, this way would lead to a satisfying logical theory of nuclear structure. Indications that it is the correct way may be found already in the thesis of A. M. Feingold (1952) (cf. Feingold & Wigner 1950). A method of calculation alternative to that described here or in Elliott's thesis is the tensor operator

† This work is being carried out with the Pease-Feshbach interaction transformed to a charge-symmetric form. It has been shown by E. H. Kronheimer (1953) that the original neutral Pease-Feshbach interaction leads to a large excess binding energy for ^{9}Be, using the single-particle s^4p^5 configuration with Gauss radial wave functions; it has been shown further by P. G. Wakely (unpublished) that the Serber-type Pease-Feshbach interaction leads to a large excess binding energy for ^{40}Ca (using single-particle Gauss radial wave functions and a multiple closed-shell configuration). The neutral, symmetric and Serber-type interactions, which are identical for the deuteron, are all three almost equivalent for the triton and α-particle. In the p-shell nuclei, however, they differ considerably (as first shown by E. H. Kronheimer for the ^{9}Be nucleus). (Cf. the results of S. F. Edwards, (1952).) It is proving by no means an easy task to obtain sufficient binding energy for these nuclei with the charge-symmetric Pease-Feshbach interaction, and probably quite a large number of single particle configurations will contribute appreciably to the ground state of each nucleus.

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method of Racah. A detailed description of this alternative method, with applications to the calculation of the odd excited states of ^{16}O, with central, tensor and spin-orbit forces and interconfigurational mixing, may be found in the thesis of J. Hope (1952).

Corrections to Part V. (Elliott, J. P. 1953 *Proc. Roy. Soc.* A, **218**, 345)

p. 351, formula (16): first U function on the left-hand side should read $U(ebgk;ah)$.

p. 351, formula (17): the summation should read $\sum_{\Phi_r \Phi'_r}$.

p. 351, formula (18): the T matrix element should read $\langle T_2 \| T \| T_2 \rangle$.

p. 363, Read 8 for δ in the formula for Q.

p. 370, appendix, formula (25): first U function on the right-hand side should read $U(j_1 j_2 J_{123} j_3; J_{12} J_{23})$.

p. 370, seven lines from bottom: in expression for T, first U function should again read $U(j_1 j_2 J_{123} j_3; J_{12} J_{23})$.

p. 370, last line but one: in expression for T, insert factor $(-1)^{j_1+J_{23}-J_{123}}$; also last U function should read $U(j_3 J j_2 J_{14}; J_{124} J_{23})$.

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TABLE 1. $\langle L'' | L \rangle_{L'L_2} = U(L''1L1; L'L_2)$

$\langle S|S \rangle$

	S
P	1

$\langle S|P \rangle$, $\langle P|S \rangle$

	P
P	1

$\langle S|D \rangle$, $\langle D|S \rangle$

	D
P	1

$\langle P|P \rangle$

	S	P	D
S	$\sqrt{\frac{1}{9}}$	$-\sqrt{\frac{3}{9}}$	$\sqrt{\frac{5}{9}}$
P	$-\sqrt{\frac{4}{12}}$	$\sqrt{\frac{3}{12}}$	$\sqrt{\frac{5}{12}}$
D	$\sqrt{\frac{20}{36}}$	$\sqrt{\frac{15}{36}}$	$\sqrt{\frac{1}{36}}$

$\langle P|D \rangle$, $\langle D|P \rangle$

	P	D
P	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
D	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$

$\langle P|F \rangle$, $\langle F|P \rangle$

	D
D	1

$\langle D|D \rangle$

	S	P	D
P	$\sqrt{\frac{4}{20}}$	$-\sqrt{\frac{9}{20}}$	$\sqrt{\frac{7}{20}}$
D	$-\sqrt{\frac{4}{12}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{7}{12}}$
F	$\sqrt{\frac{7}{15}}$	$\sqrt{\frac{7}{15}}$	$\sqrt{\frac{1}{15}}$

$\langle D|F \rangle$, $\langle F|D \rangle$

	P	D
D	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
F	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle D|G \rangle$, $\langle G|D \rangle$

	D
F	1

$\langle F|F \rangle$

	S	P	D
D	$\sqrt{\frac{5}{21}}$	$-\sqrt{\frac{10}{21}}$	$\sqrt{\frac{6}{21}}$
F	$-\sqrt{\frac{8}{24}}$	$\sqrt{\frac{1}{24}}$	$\sqrt{\frac{15}{24}}$
G	$\sqrt{\frac{24}{56}}$	$\sqrt{\frac{27}{56}}$	$\sqrt{\frac{5}{56}}$

$\langle F|G \rangle$, $\langle G|F \rangle$

	P	D
F	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{5}{8}}$
G	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{3}{8}}$

$\langle G|G \rangle$

	S	P	D
F	$\sqrt{\frac{56}{216}}$	$-\sqrt{\frac{105}{216}}$	$\sqrt{\frac{55}{216}}$
G	$-\sqrt{\frac{40}{120}}$	$\sqrt{\frac{3}{120}}$	$\sqrt{\frac{77}{120}}$

(The rows are labelled with the values of L', the columns with the values of L_2.)

Table 2a (n even). $\langle 2T+1, 2S+1 \,|\, 2T''+1, 2S''+1\rangle_{2T'+1,\, 2S'+1;\, 2T_2+1,\, 2S_2+1}$
$\equiv U(T''\tfrac{1}{2}T\tfrac{1}{2};\, T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};\, S'S_2)$

$\langle 11|11\rangle$

	11
22	1

$\langle 11|13\rangle$, $\langle 13|11\rangle$

	13
22	1

$\langle 11|31\rangle$, $\langle 31|11\rangle$

	31
22	1

$\langle 11|33\rangle$, $\langle 33|11\rangle$

	33
22	1

$\langle 13|13\rangle$

	11	13
22	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
24	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 13|31\rangle$, $\langle 31|13\rangle$

	33
22	1

$\langle 13|33\rangle$, $\langle 33|13\rangle$

	31	33
22	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
24	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 13|15\rangle$, $\langle 15|13\rangle$

	13
24	1

$\langle 13|35\rangle$, $\langle 35|13\rangle$

	33
24	1

$\langle 31|31\rangle$

	11	31
22	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
42	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 31|33\rangle$, $\langle 33|31\rangle$

	13	33
22	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
42	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 31|51\rangle$, $\langle 51|31\rangle$

	31
42	1

$\langle 31|53\rangle$, $\langle 53|31\rangle$

	33
42	1

$\langle 33|33\rangle$

	11	13	31	33
22	$\sqrt{\frac{1}{9}}$	$-\sqrt{\frac{2}{9}}$	$-\sqrt{\frac{2}{9}}$	$\sqrt{\frac{4}{9}}$
24	$-\sqrt{\frac{2}{9}}$	$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{4}{9}}$	$\sqrt{\frac{2}{9}}$
42	$-\sqrt{\frac{2}{9}}$	$\sqrt{\frac{4}{9}}$	$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{2}{9}}$
44	$\sqrt{\frac{4}{9}}$	$\sqrt{\frac{2}{9}}$	$\sqrt{\frac{2}{9}}$	$\sqrt{\frac{1}{9}}$

$\langle 33|15\rangle$, $\langle 15|33\rangle$

	33
24	1

$\langle 33|51\rangle$, $\langle 51|33\rangle$

	33
42	1

$\langle 33|35\rangle$, $\langle 35|33\rangle$

	13	33
24	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
44	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 33|53\rangle$, $\langle 53|33\rangle$

	31	33
42	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$
44	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle 15|15\rangle$

	11	13
24	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
26	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

$\langle 15|35\rangle$, $\langle 35|15\rangle$

	31	33
24	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
26	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

$\langle 15|17\rangle$, $\langle 17|15\rangle$

	13
26	1

$\langle 51|51\rangle$

	11	31
42	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
62	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

$\langle 51|53\rangle$, $\langle 53|51\rangle$

	13	33
42	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
62	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

$\langle 51|71\rangle$, $\langle 71|51\rangle$

	31
62	1

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Table 2a (contd.)

⟨35|35⟩

	11	13	31	33
24	$\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{3}{15}}$	$-\sqrt{\frac{4}{15}}$	$\sqrt{\frac{6}{15}}$
26	$-\sqrt{\frac{3}{15}}$	$-\sqrt{\frac{2}{15}}$	$\sqrt{\frac{6}{15}}$	$\sqrt{\frac{4}{15}}$
44	$-\sqrt{\frac{4}{15}}$	$\sqrt{\frac{6}{15}}$	$-\sqrt{\frac{2}{15}}$	$\sqrt{\frac{3}{15}}$

⟨35|53⟩
⟨53|35⟩

	33
44	1

⟨35|17⟩
⟨17|35⟩

	33
26	1

⟨53|53⟩

	11	31	13	33
42	$\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{3}{15}}$	$-\sqrt{\frac{4}{15}}$	$\sqrt{\frac{6}{15}}$
62	$-\sqrt{\frac{3}{15}}$	$-\sqrt{\frac{2}{15}}$	$\sqrt{\frac{6}{15}}$	$\sqrt{\frac{4}{15}}$
44	$-\sqrt{\frac{4}{15}}$	$\sqrt{\frac{6}{15}}$	$-\sqrt{\frac{2}{15}}$	$\sqrt{\frac{3}{15}}$

⟨53|71⟩
⟨71|53⟩

	33
62	1

⟨17|17⟩

	11	13
26	$-\sqrt{\frac{3}{7}}$	$\sqrt{\frac{4}{7}}$

⟨55|33⟩
⟨33|55⟩

	33
44	1

⟨71|71⟩

	11	31
62	$-\sqrt{\frac{3}{7}}$	$\sqrt{\frac{4}{7}}$

(The rows are labelled with the values of $2T'+1$, $2S'+1$, the columns with the values of $2T_2+1$, $2S_2+1$.)

Table 2b (n odd). $\langle 2T+1, 2S+1 \,|\, 2T''+1, 2S''+1\rangle_{2T'+1,\, 2S'+1;\, 2T_2+1,\, 2S_2+1}$
$\equiv U(T''\tfrac{1}{2}T\tfrac{1}{2};\, T'T_2)\,U(S''\tfrac{1}{2}S\tfrac{1}{2};\, S'S_2)$

⟨22|22⟩

	11	13	31	33
11	$\sqrt{\frac{1}{16}}$	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{3}{16}}$	$\sqrt{\frac{9}{16}}$
13	$-\sqrt{\frac{3}{16}}$	$-\sqrt{\frac{1}{16}}$	$\sqrt{\frac{9}{16}}$	$\sqrt{\frac{3}{16}}$
31	$-\sqrt{\frac{3}{16}}$	$\sqrt{\frac{9}{16}}$	$-\sqrt{\frac{1}{16}}$	$\sqrt{\frac{3}{16}}$
33	$\sqrt{\frac{9}{16}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{1}{16}}$

⟨22|24⟩
⟨24|22⟩

	13	33
13	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
33	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$

⟨22|42⟩
⟨42|22⟩

	31	33
31	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
33	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$

⟨22|44⟩
⟨44|22⟩

	33
33	1

Table 2b (contd.)

$\langle 24|24\rangle$

	11	13	31	33
13	$\sqrt{\frac{3}{32}}$	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{9}{32}}$	$\sqrt{\frac{15}{32}}$
15	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{3}{32}}$	$\sqrt{\frac{15}{32}}$	$\sqrt{\frac{9}{32}}$
33	$-\sqrt{\frac{9}{32}}$	$\sqrt{\frac{15}{32}}$	$-\sqrt{\frac{3}{32}}$	$\sqrt{\frac{5}{32}}$
35	$\sqrt{\frac{15}{32}}$	$\sqrt{\frac{9}{32}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{3}{32}}$

$\langle 24|42\rangle$ $\langle 42|24\rangle$

	33
33	1

$\langle 24|44\rangle$ $\langle 44|24\rangle$

	31	33
33	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{5}{8}}$
35	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{3}{8}}$

$\langle 24|26\rangle$ $\langle 26|24\rangle$

	13	33
15	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
35	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$

$\langle 42|42\rangle$

	11	31	13	33
31	$\sqrt{\frac{3}{32}}$	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{9}{32}}$	$\sqrt{\frac{15}{32}}$
51	$-\sqrt{\frac{5}{32}}$	$-\sqrt{\frac{3}{32}}$	$\sqrt{\frac{15}{32}}$	$\sqrt{\frac{9}{32}}$
33	$-\sqrt{\frac{9}{32}}$	$\sqrt{\frac{15}{32}}$	$-\sqrt{\frac{3}{32}}$	$\sqrt{\frac{5}{32}}$
53	$\sqrt{\frac{15}{32}}$	$\sqrt{\frac{9}{32}}$	$\sqrt{\frac{5}{32}}$	$\sqrt{\frac{3}{32}}$

$\langle 42|44\rangle$ $\langle 44|42\rangle$

	13	33
33	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{5}{8}}$
53	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{3}{8}}$

$\langle 42|62\rangle$ $\langle 62|42\rangle$

	31	23
51	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
53	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$

$\langle 44|44\rangle$

	11	13	31	33
33	$\sqrt{\frac{9}{64}}$	$-\sqrt{\frac{15}{64}}$	$-\sqrt{\frac{15}{64}}$	$\sqrt{\frac{25}{64}}$
35	$-\sqrt{\frac{15}{64}}$	$-\sqrt{\frac{9}{64}}$	$\sqrt{\frac{25}{64}}$	$\sqrt{\frac{15}{64}}$
53	$-\sqrt{\frac{15}{64}}$	$\sqrt{\frac{25}{64}}$	$-\sqrt{\frac{9}{64}}$	$\sqrt{\frac{15}{64}}$
55	$\sqrt{\frac{25}{64}}$	$\sqrt{\frac{15}{64}}$	$\sqrt{\frac{15}{64}}$	$\sqrt{\frac{9}{64}}$

$\langle 44|26\rangle$ $\langle 26|44\rangle$

	33
35	1

$\langle 44|62\rangle$ $\langle 62|44\rangle$

	33
53	1

$\langle 26|26\rangle$

	11	13	31	33
15	$\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{15}{48}}$	$\sqrt{\frac{21}{48}}$
17	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{21}{48}}$	$\sqrt{\frac{15}{48}}$
35	$-\sqrt{\frac{15}{48}}$	$\sqrt{\frac{21}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{7}{48}}$

$\langle 62|62\rangle$

	11	31	13	33
51	$\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{15}{48}}$	$\sqrt{\frac{21}{48}}$
71	$-\sqrt{\frac{7}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{21}{48}}$	$\sqrt{\frac{15}{48}}$
53	$-\sqrt{\frac{15}{48}}$	$\sqrt{\frac{21}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{7}{48}}$

(The rows are labelled with the values of $2T'+1$, $2S'+1$, the columns with the values of $2T_2+1$, $2S_2+1$.)

Table 3. $\langle n \mid n-2, 2\rangle$ weight factors

$\langle p^3 \mid p^2p\rangle$	[2]	[11]
	[1] (10) $[\tilde{1}]$ (100)	[1] (10) $[\tilde{1}]$ (100)
[3] (30) $[\tilde{3}]$ (001)	1	
[21] (11) $[\widetilde{21}]$ (110)	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
[111] (00) $[\widetilde{111}]$ (300)		1

$\langle p^4 \mid p^2p^2\rangle$	[2]		[11]	
	[2] (20) $[\tilde{2}]$ (010)	[11] (01) $[\widetilde{11}]$ (200)	[2] (20) $[\tilde{2}]$ (010)	[11] (01) $[\widetilde{11}]$ (200)
[4] (40) $[\tilde{4}]$ (000)	1			
[31] (21) $[\widetilde{31}]$ (101)	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	
[22] (02) $[\widetilde{22}]$ (020)	$\sqrt{\frac{1}{2}}$			$\sqrt{\frac{1}{2}}$
[211] (10) $[\widetilde{211}]$ (210)		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$

$\langle p^5 \mid p^3p^2\rangle$	[2]			[11]		
	[3] (30) $[\tilde{3}]$ (001)	[21] (11) $[\widetilde{21}]$ (110)	[111] (00) $[\widetilde{111}]$ (300)	[3] (30) $[\tilde{3}]$ (001)	[21] (11) $[\widetilde{21}]$ (110)	[111] (00) $[\widetilde{111}]$ (300)
[41] (31) $[\widetilde{41}]$ (100)	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$		$\sqrt{\frac{1}{4}}$		
[32] (12) $[\widetilde{32}]$ (011)	$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{2}{5}}$			$\sqrt{\frac{2}{5}}$	
[311] (20) $[\widetilde{311}]$ (201)		$\sqrt{\frac{2}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{2}{6}}$	
[221] (01) $[\widetilde{221}]$ (120)		$\sqrt{\frac{2}{5}}$			$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{5}}$

$\langle p^6 \mid p^4p^2\rangle$	[2]				[11]			
	[4] (40) $[\tilde{4}]$ (000)	[31] (21) $[\widetilde{31}]$ (101)	[22] (02) $[\widetilde{22}]$ (020)	[211] (10) $[\widetilde{211}]$ (210)	[4] (40) $[\tilde{4}]$ (000)	[31] (21) $[\widetilde{31}]$ (101)	[22] (02) $[\widetilde{22}]$ (020)	[211] (10) $[\widetilde{211}]$ (210)
[42] (22) $[\widetilde{42}]$ (010)	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{3}{9}}$	$\sqrt{\frac{2}{9}}$			$\sqrt{\frac{3}{9}}$		
[33] (03) $[\widetilde{33}]$ (002)		$\sqrt{\frac{3}{5}}$					$\sqrt{\frac{2}{5}}$	
[411] (30) $[\widetilde{411}]$ (200)		$\sqrt{\frac{3}{10}}$		$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{10}}$		
[321] (11) $[\widetilde{321}]$ (111)		$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{2}{16}}$	$\sqrt{\frac{3}{16}}$		$\sqrt{\frac{3}{16}}$	$\sqrt{\frac{2}{16}}$	$\sqrt{\frac{3}{16}}$
[222] (00) $[\widetilde{222}]$ (030)			$\sqrt{\frac{2}{5}}$					$\sqrt{\frac{3}{5}}$

$\langle p^7 \mid p^5p^2\rangle$	[2]				[11]			
	[41] (31) $[\widetilde{41}]$ (100)	[31] (12) $[\widetilde{32}]$ (011)	[311] (20) $[\widetilde{311}]$ (201)	[221] (01) $[\widetilde{221}]$ (120)	[41] (31) $[\widetilde{41}]$ (100)	[32] (12) $[\widetilde{32}]$ (011)	[311] (20) $[\widetilde{311}]$ (201)	[221] (01) $[\widetilde{221}]$ (120)
[43] (13) $[\widetilde{43}]$ (001)	$\sqrt{\frac{4}{14}}$	$\sqrt{\frac{5}{14}}$				$\sqrt{\frac{5}{14}}$		
[421] (21) $[\widetilde{421}]$ (110)	$\sqrt{\frac{4}{35}}$	$\sqrt{\frac{5}{35}}$	$\sqrt{\frac{6}{35}}$	$\sqrt{\frac{5}{35}}$	$\sqrt{\frac{4}{35}}$	$\sqrt{\frac{5}{35}}$	$\sqrt{\frac{6}{35}}$	
[331] (02) $[\widetilde{331}]$ (102)		$\sqrt{\frac{5}{21}}$	$\sqrt{\frac{6}{21}}$			$\sqrt{\frac{5}{21}}$		$\sqrt{\frac{5}{21}}$
[322] (10) $[\widetilde{322}]$ (021)		$\sqrt{\frac{5}{21}}$		$\sqrt{\frac{5}{21}}$			$\sqrt{\frac{6}{21}}$	$\sqrt{\frac{5}{21}}$

Table 3 (*cont.*)

$\langle p^8 \| p^6p^2\rangle$	[2] [42] (22) $[\widetilde{42}]$ (010)	[2] [33] (03) $[\widetilde{33}]$ (002)	[2] [411] (30) $[\widetilde{411}]$ (200)	[2] [321] (11) $[\widetilde{321}]$ (111)	[2] [222] (00) $[\widetilde{222}]$ (030)	[11] [42] (22) $[\widetilde{42}]$ (010)	[11] [33] (03) $[\widetilde{33}]$ (002)	[11] [411] (30) $[\widetilde{411}]$ (200)	[11] [321] (11) $[\widetilde{321}]$ (111)	[11] [222] (00) $[\widetilde{222}]$ (030)
[44] (04) $[\widetilde{44}]$ (000)	$\sqrt{\frac{9}{14}}$						$\sqrt{\frac{5}{14}}$			
[431] (12) $[\widetilde{431}]$ (101)	$\sqrt{\frac{9}{70}}$	$\sqrt{\frac{5}{70}}$	$\sqrt{\frac{10}{70}}$	$\sqrt{\frac{16}{70}}$		$\sqrt{\frac{9}{70}}$	$\sqrt{\frac{5}{70}}$		$\sqrt{\frac{16}{70}}$	
[422] (20) $[\widetilde{422}]$ (020)	$\sqrt{\frac{9}{56}}$			$\sqrt{\frac{16}{56}}$	$\sqrt{\frac{5}{56}}$			$\sqrt{\frac{10}{56}}$	$\sqrt{\frac{16}{56}}$	
[332] (01) $[\widetilde{332}]$ (012)		$\sqrt{\frac{5}{42}}$		$\sqrt{\frac{16}{42}}$					$\sqrt{\frac{16}{42}}$	$\sqrt{\frac{5}{42}}$

$\langle p^9 \| p^7p^2\rangle$	[2] [43] (13) $[\widetilde{43}]$ (001)	[2] [421] (21) $[\widetilde{421}]$ (110)	[2] [331] (02) $[\widetilde{331}]$ (102)	[2] [322] (10) $[\widetilde{322}]$ (021)	[11] [43] (13) $[\widetilde{43}]$ (001)	[11] [421] (21) $[\widetilde{421}]$ (110)	[11] [331] (02) $[\widetilde{331}]$ (102)	[11] [322] (10) $[\widetilde{322}]$ (021)
[441] (03) $[\widetilde{441}]$ (100)	$\sqrt{\frac{2}{12}}$	$\sqrt{\frac{5}{12}}$			$\sqrt{\frac{2}{12}}$		$\sqrt{\frac{3}{12}}$	
[432] (11) $[\widetilde{432}]$ (011)	$\sqrt{\frac{2}{24}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{3}{24}}$	$\sqrt{\frac{3}{24}}$		$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{3}{24}}$	$\sqrt{\frac{3}{24}}$
[333] (00) $[\widetilde{333}]$ (003)			$\sqrt{\frac{1}{2}}$					$\sqrt{\frac{1}{2}}$

$\langle p^{10} \| p^8p^2\rangle$	[2] [44] (04) $[\widetilde{44}]$ (000)	[2] [431] (12) $[\widetilde{431}]$ (101)	[2] [422] (20) $[\widetilde{422}]$ (020)	[2] [332] (01) $[\widetilde{332}]$ (012)	[11] [431] (12) $[\widetilde{431}]$ (101)	[11] [422] (20) $[\widetilde{422}]$ (020)	[11] [332] (01) $[\widetilde{332}]$ (012)
[442] (02) $[\widetilde{442}]$ (010)	$\sqrt{\frac{1}{18}}$	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{4}{18}}$		$\sqrt{\frac{5}{18}}$		$\sqrt{\frac{3}{18}}$
[433] (10) $[\widetilde{433}]$ (002)		$\sqrt{\frac{5}{15}}$		$\sqrt{\frac{3}{15}}$		$\sqrt{\frac{4}{15}}$	$\sqrt{\frac{3}{15}}$

$\langle p^{11} \| p^9p^2\rangle$	[2] [441] (03) $[\widetilde{441}]$ (100)	[2] [432] (11) $[\widetilde{432}]$ (011)	[11] [432] (11) $[\widetilde{432}]$ (011)	[11] [333] (00) $[\widetilde{333}]$ (003)
[443] (01) $[\widetilde{443}]$ (001)	$\sqrt{\frac{2}{11}}$	$\sqrt{\frac{4}{11}}$	$\sqrt{\frac{4}{11}}$	$\sqrt{\frac{1}{11}}$

$\langle p^{12} \| p^{10}p^2\rangle$	[2] [442] (02) $[\widetilde{442}]$ (010)	[11] [433] (10) $[\widetilde{433}]$ (002)
[444] (00) $[\widetilde{444}]$ (000)	$\sqrt{\frac{6}{11}}$	$\sqrt{\frac{5}{11}}$

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Table 4. $\langle p^n | p^{n-2}p^2 \rangle$ orbital coefficients

[111], [222], [333], [444]		[1], [211], [322] (10)	[22], [331], [442] (02)	
		PP	*SS*	*DD*
(00)	S*	1	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{5}{6}}$

[3], [411]		[1], [211], [322] (10)		[4] (40)			[31] (21)							
		PS	*PD*	*SP*	*DP*	*GP*	*PP*	*DP*	*FP*	*PS*	*FS*	*PD*	*DD*	*FD*
(30)	*P*	$\sqrt{\frac{5}{9}}$	$\sqrt{\frac{4}{9}}$	$\sqrt{\frac{2}{9}}$	$\sqrt{\frac{7}{9}}$		$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$		$\sqrt{\frac{5}{135}}$		$\sqrt{\frac{64}{135}}$	$\sqrt{\frac{10}{135}}$	$\sqrt{\frac{56}{135}}$
	F		1		$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{6}{7}}$		$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$		$\sqrt{\frac{10}{45}}$	$-\sqrt{\frac{3}{45}}$	$\sqrt{\frac{20}{45}}$	$\sqrt{\frac{12}{45}}$

[21], [321], [432]		[1], [211], [322] (10)			[22], [331] (02)					[31], [421] (21)									[43] (13)					
		PP	*PS*	*PD*	*SP*	*DP*	*DS*	*SD*	*DD*	*PP*	*DP*	*FP*	*PS*	*DS*	*PD*	*DD*	*FD*	*PS*	*DS*	*PD*	*DD*	*FD*	*GD*
(11)	*P*	1	$-\sqrt{\frac{4}{9}}$	$\sqrt{\frac{5}{9}}$	$-\sqrt{\frac{4}{9}}$	$\sqrt{\frac{5}{9}}$			-1	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$		$\sqrt{\frac{20}{135}}$		$\sqrt{\frac{49}{135}}$	$-\sqrt{\frac{10}{135}}$	$-\sqrt{\frac{56}{135}}$	$\sqrt{\frac{5}{27}}$		$-\sqrt{\frac{1}{27}}$	$\sqrt{\frac{7}{27}}$	$\sqrt{\frac{14}{27}}$	
	*D**	-1		1		1	$-\sqrt{\frac{4}{15}}$	$-\sqrt{\frac{4}{15}}$	$-\sqrt{\frac{7}{15}}$	$-\sqrt{\frac{9}{75}}$	$\sqrt{\frac{10}{75}}$	$-\sqrt{\frac{56}{75}}$		$-\sqrt{\frac{40}{225}}$	$-\sqrt{\frac{3}{225}}$	$\sqrt{\frac{70}{225}}$	$\sqrt{\frac{112}{225}}$		$-\sqrt{\frac{7}{45}}$	$\sqrt{\frac{3}{45}}$	$-\sqrt{\frac{1}{45}}$	$\sqrt{\frac{7}{45}}$	$-\sqrt{\frac{27}{45}}$

[4]		[2] (20)			
		SS	*DS*	*SD*	*DD*
	S	$\sqrt{\frac{5}{9}}$			$\sqrt{\frac{4}{9}}$
(40)	*D*		$\sqrt{\frac{7}{18}}$	$\sqrt{\frac{7}{18}}$	$\sqrt{\frac{4}{18}}$
	G				1

[22], [331], [442]		[11], [221], [332] (01)	[2], [311], [422] (20)				[32], [431] (12)							[44] (04)				
		PP	*SS*	*DS*	*SD*	*DD*	*PP*	*DP*	*FP*	*DS*	*PD*	*DD*	*FD*	*SS*	*DS*	*SD*	*DD*	*GD*
(02)	*S*	-1	$-\sqrt{\frac{4}{9}}$			$\sqrt{\frac{5}{9}}$	1					1		$\sqrt{\frac{2}{9}}$			$\sqrt{\frac{7}{9}}$	
	D	-1		$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{1}{9}}$	$-\sqrt{\frac{7}{9}}$	$-\sqrt{\frac{1}{25}}$	$\sqrt{\frac{10}{25}}$	$\sqrt{\frac{14}{25}}$	$-\sqrt{\frac{5}{25}}$	$-\sqrt{\frac{6}{25}}$	·	$-\sqrt{\frac{14}{25}}$		$\sqrt{\frac{35}{225}}$	$\sqrt{\frac{8}{225}}$	$\sqrt{\frac{20}{225}}$	$\sqrt{\frac{162}{225}}$

[211], [322], [433]		[11], [221], [332] (01)			[2], [311], [422] (20)		[32], [431] (12)			
		PP	*PS*	*PD*	*SP*	*DP*	*PS*	*PD*	*DD*	*FD*
(10)	*P*	-1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{5}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{5}{6}}$	$\sqrt{\frac{5}{30}}$	$-\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{10}{30}}$	$\sqrt{\frac{14}{30}}$

[31], [421]		[2], [311] (20)					[11], [221] (01)		[32] (12)								
		SP	*DP*	*DS*	*SD*	*DD*	*PS*	*PD*	*PP*	*DP*	*FP*	*PS*	*DS*	*FS*	*PD*	*DD*	*FD*
	P	$-\sqrt{\frac{5}{6}}$	$\sqrt{\frac{1}{6}}$			1	$\sqrt{\frac{5}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$		$-\sqrt{\frac{5}{270}}$			$\sqrt{\frac{49}{270}}$	$\sqrt{\frac{90}{270}}$	$\sqrt{\frac{126}{270}}$
(21)	*D*		1	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	·		1	$\sqrt{\frac{18}{45}}$	$\sqrt{\frac{20}{45}}$	$-\sqrt{\frac{7}{45}}$		$-\sqrt{\frac{10}{135}}$		$-\sqrt{\frac{27}{135}}$	$-\sqrt{\frac{70}{135}}$	$\sqrt{\frac{28}{135}}$
	F		-1			1		1		$\sqrt{\frac{1}{9}}$	$-\sqrt{\frac{8}{9}}$			$\sqrt{\frac{40}{135}}$	$\sqrt{\frac{27}{135}}$	$-\sqrt{\frac{20}{135}}$	$\sqrt{\frac{48}{135}}$

[31], [421]		[41] (31)										
		PP	*DP*	*FP*	*GP*	*PS*	*DS*	*FS*	*PD*	*DD*	*FD*	*GD*
	P	$\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{7}{12}}$			$-\sqrt{\frac{20}{108}}$			$\sqrt{\frac{25}{108}}$	$\sqrt{\frac{63}{108}}$	·	
(21)	*D*	$-\sqrt{\frac{9}{72}}$	$-\sqrt{\frac{7}{72}}$	$-\sqrt{\frac{56}{72}}$			$\sqrt{\frac{28}{216}}$		$\sqrt{\frac{27}{216}}$	$\sqrt{\frac{25}{216}}$	$-\sqrt{\frac{28}{216}}$	$-\sqrt{\frac{108}{216}}$
	F		$-\sqrt{\frac{64}{504}}$	$\sqrt{\frac{35}{504}}$	$-\sqrt{\frac{405}{504}}$			$-\sqrt{\frac{280}{1512}}$	·	$-\sqrt{\frac{32}{1512}}$	$\sqrt{\frac{525}{1512}}$	$\sqrt{\frac{675}{1512}}$

Table 4 (*cont.*)

[32], [431]		[21], [321] (11)						[3], [411] (30)				[33] (03)					
		PP	D^*P	PS	D^*S	PD	D^*D	PS	FS	PD	FD	PP	FP	PS	FS	PD	FD
	P	$-\sqrt{\frac{5}{8}}$	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{20}{72}}$		$\sqrt{\frac{49}{72}}$	$-\sqrt{\frac{3}{72}}$	$\sqrt{\frac{20}{45}}$		$-\sqrt{\frac{4}{45}}$	$-\sqrt{\frac{21}{45}}$	-1		$\sqrt{\frac{5}{90}}$		$\sqrt{\frac{64}{90}}$	$-\sqrt{\frac{21}{90}}$
(12)	D	$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$		$\sqrt{\frac{4}{12}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{7}{12}}$			$-\sqrt{\frac{8}{15}}$	$\sqrt{\frac{7}{15}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{7}{10}}$			$-\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{14}{15}}$
	F		-1			$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$		$-\sqrt{\frac{5}{30}}$	$-\sqrt{\frac{1}{30}}$	$\sqrt{\frac{24}{30}}$		-1		$\sqrt{\frac{5}{15}}$	$\sqrt{\frac{4}{15}}$	$\sqrt{\frac{6}{15}}$

		[42] (22)												
		S^*P	$D_I P$	$D_{II}^* P$	FP	G^*P	$D_I S$	$D_{II}^* S$	FS	S^*D	$D_I D$	$D_{II}^* D$	FD	G^*D
	P	$-\sqrt{\frac{5}{27}}$	$\sqrt{\frac{15}{27}}$	$\sqrt{\frac{7}{27}}$							$-\sqrt{\frac{5}{54}}$	$\sqrt{\frac{21}{54}}$	$\sqrt{\frac{28}{54}}$	
(12)	D		$\sqrt{\frac{5}{54}}$	$-\sqrt{\frac{21}{54}}$	$\sqrt{\frac{28}{54}}$		$-\sqrt{\frac{5}{108}}$	$\sqrt{\frac{21}{108}}$		$-\sqrt{\frac{12}{108}}$	$-\sqrt{\frac{35}{108}}$	$-\sqrt{\frac{3}{108}}$	$-\sqrt{\frac{14}{108}}$	$-\sqrt{\frac{18}{108}}$
	F		$-\sqrt{\frac{35}{378}}$	$\sqrt{\frac{3}{378}}$	$\sqrt{\frac{70}{378}}$	$\sqrt{\frac{270}{378}}$			$-\sqrt{\frac{70}{378}}$		$-\sqrt{\frac{35}{378}}$	$-\sqrt{\frac{27}{378}}$	$\sqrt{\frac{21}{378}}$	$-\sqrt{\frac{225}{378}}$

[41]		[3] (30)						[21] (11)			
		PP	FP	PS	FS	PD	FD	PS	D^*S	PD	D^*D
	P	1		$-\sqrt{\frac{20}{225}}$		$\sqrt{\frac{121}{225}}$	$-\sqrt{\frac{84}{225}}$	$\sqrt{\frac{5}{9}}$		$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{3}{9}}$
(31)	D	$-\sqrt{\frac{21}{25}}$	$\sqrt{\frac{4}{25}}$			$\sqrt{\frac{7}{15}}$	$\sqrt{\frac{8}{15}}$		$\sqrt{\frac{7}{15}}$	$-\sqrt{\frac{7}{15}}$	$\sqrt{\frac{1}{15}}$
	F		1		$-\sqrt{\frac{40}{75}}$	$\sqrt{\frac{32}{75}}$	$-\sqrt{\frac{3}{75}}$			$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$
	G		-1				1				1

[311], [422]		[3], [411] (30)		[21], [321] (11)					[111], [222] (00)		[42] (22)							
		PP	FP	PP	D^*P	D^*S	PD	D^*D	S^*S	S^*D	S^*S	$D_I S$	$D_{II}^* S$	S^*D	$D_I D$	$D_{II}^* D$	FD	G^*D
(20)	S	1		1				1	1		$\sqrt{\frac{5}{27}}$				$\sqrt{\frac{15}{27}}$	$-\sqrt{\frac{7}{27}}$		
	D	$\sqrt{\frac{4}{25}}$	$\sqrt{\frac{21}{25}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{20}}$	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{7}{20}}$		1		$-\sqrt{\frac{15}{135}}$	$-\sqrt{\frac{7}{135}}$	$-\sqrt{\frac{1}{135}}$	·	$\sqrt{\frac{16}{135}}$	$\sqrt{\frac{42}{135}}$	$-\sqrt{\frac{54}{135}}$

[221], [332], [443]		[111], [222], [333] (00)	[21], [321], [432] (11)					[33], [441] (03)		
		S^*P	PP	D^*P	PS	PD	D^*D	PS	PD	FD
(01)	P	1	$\sqrt{\frac{3}{8}}$	$-\sqrt{\frac{5}{8}}$	$-\sqrt{\frac{4}{24}}$	$\sqrt{\frac{5}{24}}$	$\sqrt{\frac{15}{24}}$	$\sqrt{\frac{5}{30}}$	$\sqrt{\frac{4}{30}}$	$\sqrt{\frac{21}{30}}$

Table 4 (*cont.*)

[42]	[31] (21)								[22] (02)				[4] (40)					
	PP	*DP*	*FP*	*DS*	*FS*	*PD*	*DD*	*FD*	*SS*	*DS*	*SD*	*DD*	*SS*	*DS*	*GS*	*SD*	*DD*	*GD*
S^*	-1						1		$-\sqrt{\frac{5}{6}}$			$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{8}{15}}$				$-\sqrt{\frac{7}{15}}$	
D_{I}	$-\sqrt{\frac{18}{30}}$	$\sqrt{\frac{5}{30}}$	$\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{5}{60}}$		$\sqrt{\frac{6}{60}}$	$-\sqrt{\frac{35}{60}}$	$\sqrt{\frac{14}{60}}$		$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	·		$-\sqrt{\frac{35}{100}}$		$-\sqrt{\frac{2}{100}}$	$\sqrt{\frac{45}{100}}$	$\sqrt{\frac{18}{100}}$
(22) D^*_{II}	$\sqrt{\frac{14}{50}}$	$\sqrt{\frac{35}{50}}$	$\sqrt{\frac{1}{50}}$	$-\sqrt{\frac{35}{100}}$		$\sqrt{\frac{42}{100}}$	$\sqrt{\frac{5}{100}}$	$-\sqrt{\frac{18}{100}}$		$\sqrt{\frac{7}{30}}$	$\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{16}{30}}$		$-\sqrt{\frac{35}{420}}$		$\sqrt{\frac{98}{420}}$	$-\sqrt{\frac{125}{420}}$	$\sqrt{\frac{162}{420}}$
F		$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{10}{30}}$	$\sqrt{\frac{12}{30}}$	$\sqrt{\frac{5}{30}}$	$\sqrt{\frac{3}{30}}$				-1					$\sqrt{\frac{5}{14}}$	$-\sqrt{\frac{9}{14}}$
G^*			1				$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{5}{6}}$				1			$\sqrt{\frac{14}{70}}$		$\sqrt{\frac{1}{70}}$	$-\sqrt{\frac{55}{70}}$

[33], [441]	[22], [331] (02)		[31], [421] (21)					[43] (13)									
	SP	*DP*	*PS*	*FS*	*PD*	*DD*	*FD*	*PP*	*DP*	*FP*	*GP*	*PS*	*FS*	*PD*	*DD*	*FD*	*GD*
(03) P	$-\sqrt{\frac{5}{9}}$	$-\sqrt{\frac{4}{9}}$	$\sqrt{\frac{40}{135}}$		$-\sqrt{\frac{8}{135}}$	$\sqrt{\frac{80}{135}}$	$-\sqrt{\frac{7}{135}}$	$\sqrt{\frac{5}{12}}$	$\sqrt{\frac{7}{12}}$			$-\sqrt{\frac{20}{540}}$		$\sqrt{\frac{121}{540}}$	$-\sqrt{\frac{175}{540}}$	$\sqrt{\frac{224}{540}}$	
F		-1		$-\sqrt{\frac{5}{45}}$	$-\sqrt{\frac{6}{45}}$	$-\sqrt{\frac{10}{45}}$	$\sqrt{\frac{24}{45}}$		$-\sqrt{\frac{4}{84}}$	$\sqrt{\frac{35}{84}}$	$\sqrt{\frac{45}{84}}$		$-\sqrt{\frac{280}{1260}}$	$-\sqrt{\frac{84}{1260}}$	$-\sqrt{\frac{200}{1260}}$	$-\sqrt{\frac{21}{1260}}$	$-\sqrt{\frac{675}{1260}}$

[43]	[32] (12)									[41] (31)							
	PP	*DP*	*FP*	*PS*	*DS*	*FS*	*PD*	*DD*	*FD*	*PS*	*DS*	*FS*	*GS*	*PD*	*DD*	*FD*	*GD*
P	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{8}{27}}$			$\sqrt{\frac{10}{27}}$	$-\sqrt{\frac{9}{27}}$	·	$\sqrt{\frac{40}{135}}$				$\sqrt{\frac{32}{135}}$	·	$-\sqrt{\frac{63}{135}}$	
(13) D	$\sqrt{\frac{126}{225}}$	$-\sqrt{\frac{35}{225}}$	$\sqrt{\frac{64}{225}}$		$\sqrt{\frac{140}{675}}$		$-\sqrt{\frac{378}{675}}$	$\sqrt{\frac{125}{675}}$	$\sqrt{\frac{32}{675}}$		$-\sqrt{\frac{112}{378}}$			·	$\sqrt{\frac{64}{378}}$	$-\sqrt{\frac{175}{378}}$	$\sqrt{\frac{27}{378}}$
F		$\sqrt{\frac{8}{9}}$	$\sqrt{\frac{1}{9}}$			$-\sqrt{\frac{8}{27}}$	·	$\sqrt{\frac{4}{27}}$	$\sqrt{\frac{15}{27}}$			$-\sqrt{\frac{70}{3780}}$		$-\sqrt{\frac{756}{3780}}$	$\sqrt{\frac{1250}{3780}}$	$\sqrt{\frac{1029}{3780}}$	$-\sqrt{\frac{675}{3780}}$
G			1					$-\sqrt{\frac{4}{9}}$	$-\sqrt{\frac{5}{9}}$				$\sqrt{\frac{42}{252}}$		$\sqrt{\frac{10}{252}}$	$\sqrt{\frac{35}{252}}$	$-\sqrt{\frac{165}{252}}$

[44]	[33] (03)		[42] (22)								
	PP	*FP*	S^*S	$D_{\mathrm{I}}S$	$D^*_{\mathrm{II}}S$	G^*S	S^*D	$D_{\mathrm{I}}D$	$D^*_{\mathrm{II}}D$	*FD*	G^*D
S	-1		$-\sqrt{\frac{16}{54}}$					$-\sqrt{\frac{3}{54}}$	$-\sqrt{\frac{35}{54}}$		
(04) D	$-\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{3}{10}}$		$-\sqrt{\frac{735}{3780}}$	$\sqrt{\frac{175}{3780}}$		$\sqrt{\frac{196}{3780}}$	$\sqrt{\frac{945}{3780}}$	$\sqrt{\frac{625}{3780}}$	$-\sqrt{\frac{1050}{3780}}$	$-\sqrt{\frac{54}{3780}}$
G		-1				$-\sqrt{\frac{14}{126}}$		$\sqrt{\frac{7}{126}}$	$-\sqrt{\frac{15}{126}}$	$\sqrt{\frac{35}{126}}$	$\sqrt{\frac{55}{126}}$

Table 5a. $\langle \gamma^n | \gamma^{n-2}\gamma^2 \rangle$ charge-spin coefficients (n even)

$[\tilde{4}], [\widetilde{44}], [\widetilde{444}]$		$[\tilde{2}], [\widetilde{42}], [\widetilde{442}]$ (010)		$[\widetilde{33}], [\widetilde{433}]$ (002)	
		13* 13	31 31	11 11	33 33
(000)	11	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{9}{10}}$

$[\widetilde{211}]$		$[\widetilde{11}]$ (200)							$[\tilde{2}]$ (010)			
		11 13	33 13	11 31	33 31	33 11	11 33	33 33	13* 11	31 11	13* 33	31 33
	13	$\sqrt{\frac{3}{4}}$			$-\sqrt{\frac{1}{4}}$			1	$\sqrt{\frac{3}{4}}$			$-\sqrt{\frac{1}{4}}$
	31		$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$				1		$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$	
(210)	33		$-\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	.			$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
	35		1					1			1	
	53				1			1				1

$[\widetilde{31}], [\widetilde{431}]$		$[\tilde{2}], [\widetilde{42}]$ (010)								$[\widetilde{11}], [\widetilde{411}]$ (200)				$[\widetilde{33}]$ (002)						
		13* 13	31 13	13* 31	31 31	13* 11	31 11	13* 33	31 33	11 13	33 13	11 31	33 31	11 13	33 13	11 31	33 31	33 11	11 33	33 33
	13	1				$\sqrt{\frac{1}{4}}$			$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$			$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$			$-\sqrt{\frac{3}{4}}$			-1
(101)	31				-1		$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{3}{4}}$			$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$			$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$				-1
	33*		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$				$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{4}{6}}$

		$[\widetilde{321}]$ (111)											
		13* 13	31 13	33_I 13	33^*_{II} 13	15* 13	35 13	13* 31	31 31	33_I 31	33^*_{II} 31	51 31	53* 31
	13	$-\sqrt{\frac{9}{64}}$				$-\sqrt{\frac{25}{64}}$				$\sqrt{\frac{30}{64}}$	.		
(101)	31			.	$\sqrt{\frac{30}{64}}$				$-\sqrt{\frac{9}{64}}$			$-\sqrt{\frac{25}{64}}$	
	33*		$-\sqrt{\frac{4}{128}}$	$-\sqrt{\frac{10}{128}}$	.		$-\sqrt{\frac{50}{128}}$	$-\sqrt{\frac{4}{128}}$		.	$-\sqrt{\frac{10}{128}}$		$-\sqrt{\frac{50}{128}}$
		13* 11	31 11	33_I 11	33^*_{II} 11	13* 33	31 33	33_I 33	33^*_{II} 33	15* 33	51 33	35 33	53* 33
	13	$-\sqrt{\frac{12}{128}}$					$-\sqrt{\frac{16}{128}}$	$\sqrt{\frac{10}{128}}$	$-\sqrt{\frac{40}{128}}$			$\sqrt{\frac{50}{128}}$	
(101)	31		$-\sqrt{\frac{12}{128}}$			$-\sqrt{\frac{16}{128}}$		$-\sqrt{\frac{40}{128}}$	$\sqrt{\frac{10}{128}}$				$\sqrt{\frac{50}{128}}$
	33*			$-\sqrt{\frac{10}{192}}$	$-\sqrt{\frac{10}{192}}$	$\sqrt{\frac{1}{192}}$	$\sqrt{\frac{1}{192}}$	$\sqrt{\frac{10}{192}}$	$\sqrt{\frac{10}{192}}$	$-\sqrt{\frac{25}{192}}$	$-\sqrt{\frac{25}{192}}$	$-\sqrt{\frac{50}{192}}$	$-\sqrt{\frac{50}{192}}$

Table 5*a* (*cont.*)

$[\tilde{2}\tilde{2}], [\widetilde{422}]$		$[\tilde{2}], [4\tilde{2}]$ (010)				$[\tilde{1}\tilde{1}], [\widetilde{411}]$ (200)				$[\widetilde{222}]$ (030)									
		13* 13	31 13	13* 31	31 31	11 11	33 11	11 33	33 33	13* 13	31 13	35 13	53* 13	17* 13	13* 31	31 31	35 31	53* 31	71 31
	11	$\sqrt{\frac{1}{2}}$			$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$			$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						$\sqrt{\frac{1}{2}}$			
(020)	33*		$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$			$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{4}{6}}$		$\sqrt{\frac{1}{18}}$	$-\sqrt{\frac{8}{18}}$			$-\sqrt{\frac{1}{18}}$			$\sqrt{\frac{8}{18}}$	
	15	1							-1	$-\sqrt{\frac{1}{25}}$				$\sqrt{\frac{14}{25}}$			$-\sqrt{\frac{10}{25}}$		
	51				1				-1				$\sqrt{\frac{10}{25}}$			$\sqrt{\frac{1}{25}}$			$-\sqrt{\frac{14}{25}}$

		$[\widetilde{321}]$ (111)													
		13* 13	31 13	33_{I} 13	33^*_{II} 13	15* 13	35 13	53* 13	13* 31	31 31	33_{I} 31	33^*_{II} 31	51 31	35 31	53* 31
	11	$-\sqrt{\frac{1}{2}}$								$\sqrt{\frac{1}{2}}$					
(020)	33*		$-\sqrt{\frac{4}{288}}$	$\sqrt{\frac{90}{288}}$			$-\sqrt{\frac{50}{288}}$		$\sqrt{\frac{4}{288}}$			$-\sqrt{\frac{90}{288}}$			$\sqrt{\frac{50}{288}}$
	15	$\sqrt{\frac{1}{16}}$				$-\sqrt{\frac{5}{16}}$								$\sqrt{\frac{10}{16}}$	
	51							$-\sqrt{\frac{10}{16}}$		$-\sqrt{\frac{1}{16}}$			$\sqrt{\frac{5}{16}}$		

		33_{I} 11	33^*_{II} 11	15* 11	51 11	13* 33	31 33	33_{I} 33	33^*_{II} 33	15* 33	51 33	35 33	53* 33
	11							$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$				
(020)	33*	$\sqrt{\frac{2}{48}}$	$-\sqrt{\frac{2}{48}}$			$\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{2}{48}}$	$-\sqrt{\frac{2}{48}}$	$-\sqrt{\frac{5}{48}}$	$\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{10}{48}}$	$\sqrt{\frac{10}{48}}$
	15			$-\sqrt{\frac{4}{32}}$				$-\sqrt{\frac{2}{32}}$	$-\sqrt{\frac{8}{32}}$			$\sqrt{\frac{18}{32}}$	
	51				$\sqrt{\frac{4}{32}}$			$\sqrt{\frac{8}{32}}$	$\sqrt{\frac{2}{32}}$				$-\sqrt{\frac{18}{32}}$

$[4\tilde{2}], [\widetilde{442}]$		$[\tilde{4}], [4\tilde{4}]$ (000)		$[3\tilde{1}], [\widetilde{431}]$ (101)								
		11 13	11 31	13 13	33* 13	31 31	33* 31	13 11	31 11	13 33	31 33	33* 33
(010)	13*	1		$-\sqrt{\frac{2}{5}}$			$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{1}{10}}$			$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{6}{10}}$
	31		1		$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$			$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{10}}$		$\sqrt{\frac{6}{10}}$

		$[\tilde{2}\tilde{2}], [\widetilde{422}]$ (020)						$[\widetilde{332}]$ (012)						
		11 13	33* 13	15 13	11 31	33* 31	51 31	13 11	31 11	13 33	31 33	33 33	35 33	53 33
(010)	13*	$\sqrt{\frac{1}{20}}$		$\sqrt{\frac{10}{20}}$		$\sqrt{\frac{9}{20}}$		$-\sqrt{\frac{3}{30}}$			$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{6}{30}}$	$-\sqrt{\frac{20}{30}}$	
	31		$-\sqrt{\frac{9}{20}}$		$-\sqrt{\frac{1}{20}}$		$-\sqrt{\frac{10}{20}}$		$\sqrt{\frac{3}{30}}$	$-\sqrt{\frac{1}{30}}$		$-\sqrt{\frac{6}{30}}$		$\sqrt{\frac{20}{30}}$

$[\widetilde{33}]$, $[\widetilde{433}]$		$[\widetilde{31}]$, $[\widetilde{431}]$ (101) 1313	3113	33* 13	1331	3131	33* 31	$[\widetilde{22}]$, $[\widetilde{422}]$ (020) 1111	33* 11	1133	33* 33	1533	5133
(002)	11	$\sqrt{\frac{1}{2}}$				$-\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{4}}$			$\sqrt{\frac{3}{4}}$		
	33		$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{2}{6}}$	$\sqrt{\frac{1}{6}}$		$-\sqrt{\frac{2}{6}}$		$\sqrt{\frac{3}{36}}$	$\sqrt{\frac{1}{36}}$	$-\sqrt{\frac{12}{36}}$	$-\sqrt{\frac{10}{36}}$	$-\sqrt{\frac{10}{36}}$

		$[\widetilde{332}]$ (012) 1313	3113	3313	3513	1331	3131	3331	5331	3311	1333	3133	3333	3533	5333
(002)	11	$-\sqrt{\frac{1}{2}}$					$-\sqrt{\frac{1}{2}}$						1		
	33		$\sqrt{\frac{1}{54}}$	$-\sqrt{\frac{6}{54}}$	$-\sqrt{\frac{20}{54}}$	$\sqrt{\frac{1}{54}}$		$-\sqrt{\frac{6}{54}}$	$-\sqrt{\frac{20}{54}}$	$-\sqrt{\frac{3}{27}}$	$-\sqrt{\frac{2}{27}}$	$-\sqrt{\frac{2}{27}}$	.	$-\sqrt{\frac{10}{27}}$	$-\sqrt{\frac{10}{27}}$

$[\widetilde{411}]$		$[\tilde{4}]$ (000) 1111	1133	$[\tilde{31}]$ (101) 1313	3113	33* 13	1331	3131	33* 31	33* 11	1333	3133	33* 33
(200)	11	1		$-\sqrt{\frac{1}{2}}$				$-\sqrt{\frac{1}{2}}$					1
	33		1		$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{2}{6}}$	$\sqrt{\frac{1}{6}}$		$\sqrt{\frac{2}{6}}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{2}{9}}$	$\sqrt{\frac{2}{9}}$	$\sqrt{\frac{4}{9}}$

		$[\widetilde{211}]$ (210) 1313	3113	3313	3513	1331	3131	3331	5331
(200)	11	$\sqrt{\frac{1}{2}}$					$-\sqrt{\frac{1}{2}}$		
	33		$-\sqrt{\frac{1}{54}}$	$\sqrt{\frac{6}{54}}$	$\sqrt{\frac{20}{54}}$	$\sqrt{\frac{1}{54}}$		$-\sqrt{\frac{6}{54}}$	$-\sqrt{\frac{20}{54}}$

$[\widetilde{321}]$		$[\widetilde{211}]$ (210) 1313	3113	3313	3513	5313	1331	3131	3331	3531	5331
(111)	13*	$-\sqrt{\frac{1}{3}}$							$-\sqrt{\frac{2}{3}}$		
	31			$-\sqrt{\frac{2}{3}}$				$-\sqrt{\frac{1}{3}}$			
	33_{I}		$-\sqrt{\frac{32}{135}}$	$-\sqrt{\frac{3}{135}}$	$-\sqrt{\frac{10}{135}}$		$\sqrt{\frac{2}{135}}$		$\sqrt{\frac{48}{135}}$		$-\sqrt{\frac{40}{135}}$
	33^*_{II}		$\sqrt{\frac{2}{135}}$	$\sqrt{\frac{48}{135}}$	$-\sqrt{\frac{40}{135}}$		$-\sqrt{\frac{32}{135}}$		$-\sqrt{\frac{3}{135}}$		$-\sqrt{\frac{10}{135}}$
	15*	$-\sqrt{\frac{1}{5}}$								$-\sqrt{\frac{4}{5}}$	
	51					$-\sqrt{\frac{4}{5}}$		$-\sqrt{\frac{1}{5}}$			
	35			$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{6}{15}}$					$\sqrt{\frac{8}{15}}$	
	53*					$\sqrt{\frac{8}{15}}$			$\sqrt{\frac{1}{15}}$		$-\sqrt{\frac{6}{15}}$
		1311	3111	3311	3511	5311	1333	3133	3333	3533	5333
(111)	13*	$-\sqrt{\frac{6}{27}}$						$-\sqrt{\frac{8}{27}}$	$\sqrt{\frac{3}{27}}$	$\sqrt{\frac{10}{27}}$	
	31		$-\sqrt{\frac{6}{27}}$				$-\sqrt{\frac{8}{27}}$		$\sqrt{\frac{3}{27}}$		$\sqrt{\frac{10}{27}}$
	33_{I}			$\sqrt{\frac{6}{45}}$			$\sqrt{\frac{9}{45}}$	$-\sqrt{\frac{4}{45}}$	$-\sqrt{\frac{6}{45}}$	$-\sqrt{\frac{20}{45}}$	.
	33^*_{II}			$\sqrt{\frac{6}{45}}$			$-\sqrt{\frac{4}{45}}$	$\sqrt{\frac{9}{45}}$	$-\sqrt{\frac{6}{45}}$	.	$-\sqrt{\frac{20}{45}}$
	15*								$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$	
	51								$-\sqrt{\frac{3}{5}}$		$-\sqrt{\frac{2}{5}}$
	35				$-\sqrt{\frac{12}{135}}$		$-\sqrt{\frac{1}{135}}$		$\sqrt{\frac{6}{135}}$	$\sqrt{\frac{36}{135}}$	$-\sqrt{\frac{80}{135}}$
	53*					$-\sqrt{\frac{12}{135}}$		$-\sqrt{\frac{1}{135}}$	$\sqrt{\frac{6}{135}}$	$-\sqrt{\frac{80}{135}}$	$\sqrt{\frac{36}{135}}$

Table 5a (cont.)

$[\widetilde{321}]$		$[\widetilde{31}]$ (101) 1313	3113	33* 13	1331	3131	33* 31	1311	3111	33* 11	1333	3133	33* 33
	13*	$\sqrt{\frac{3}{5}}$					$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{6}{15}}$				$\sqrt{\frac{8}{15}}$	$-\sqrt{\frac{1}{15}}$
	31			$-\sqrt{\frac{2}{5}}$		$-\sqrt{\frac{3}{5}}$			$-\sqrt{\frac{6}{15}}$		$-\sqrt{\frac{8}{15}}$		$\sqrt{\frac{1}{15}}$
	33_{I}		.	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$		.			$\sqrt{\frac{2}{9}}$	$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{4}{9}}$	$-\sqrt{\frac{2}{9}}$
(111)	33^*_{II}		$\sqrt{\frac{2}{3}}$	.	.		$-\sqrt{\frac{1}{3}}$			$-\sqrt{\frac{2}{9}}$	$-\sqrt{\frac{4}{9}}$	$\sqrt{\frac{1}{9}}$	$\sqrt{\frac{2}{9}}$
	15*	-1											-1
	51					1							1
	35			-1							$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{2}{3}}$
	53*						1					$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$

		$[\widetilde{22}]$ (020) 1113	33* 13	1513	5113	1131	33* 31	1531	5131	33* 11	1511	5111	1133	33* 33	1533	5133
	13*	$-\sqrt{\frac{8}{15}}$		$\sqrt{\frac{5}{15}}$			$-\sqrt{\frac{2}{15}}$							-1		
	31		$-\sqrt{\frac{2}{15}}$			$-\sqrt{\frac{8}{15}}$			$\sqrt{\frac{5}{15}}$					-1		
	33_{I}		-1				.			$-\sqrt{\frac{6}{45}}$			$\sqrt{\frac{8}{45}}$	$-\sqrt{\frac{6}{45}}$	$-\sqrt{\frac{5}{45}}$	$\sqrt{\frac{20}{45}}$
(111)	33^*_{II}		.				-1			$-\sqrt{\frac{6}{45}}$			$\sqrt{\frac{8}{45}}$	$-\sqrt{\frac{6}{45}}$	$\sqrt{\frac{20}{45}}$	$-\sqrt{\frac{5}{45}}$
	15*			1							$\sqrt{\frac{2}{5}}$			$-\sqrt{\frac{3}{5}}$		
	51								1			$\sqrt{\frac{2}{5}}$		$-\sqrt{\frac{3}{5}}$		
	35		$-\sqrt{\frac{1}{3}}$					$-\sqrt{\frac{2}{3}}$						$-\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{5}}$	
	53*				$-\sqrt{\frac{2}{3}}$		$-\sqrt{\frac{1}{3}}$							$-\sqrt{\frac{2}{5}}$		$-\sqrt{\frac{3}{5}}$

$[\widetilde{222}]$		$[\widetilde{22}]$ (020) 1113	33* 13	1513	5113	1131	33* 31	1531	5131	$[\widetilde{211}]$ (210) 1311	3111	3511	5311	1333	3133	3333	3533	5333
	13*	$-\sqrt{\frac{5}{12}}$		$-\sqrt{\frac{2}{12}}$			$\sqrt{\frac{5}{12}}$			$\sqrt{\frac{15}{54}}$					$-\sqrt{\frac{5}{54}}$	$\sqrt{\frac{30}{54}}$	$-\sqrt{\frac{4}{54}}$	
	31		$\sqrt{\frac{5}{12}}$			$-\sqrt{\frac{5}{12}}$			$-\sqrt{\frac{2}{12}}$		$\sqrt{\frac{15}{54}}$			$-\sqrt{\frac{5}{54}}$		$\sqrt{\frac{30}{54}}$		$-\sqrt{\frac{4}{54}}$
(030)	35		$-\sqrt{\frac{2}{3}}$					$\sqrt{\frac{1}{3}}$				$\sqrt{\frac{3}{27}}$		$\sqrt{\frac{4}{27}}$		$\sqrt{\frac{6}{27}}$	$-\sqrt{\frac{9}{27}}$	$-\sqrt{\frac{5}{27}}$
	53*				$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{2}{3}}$						$\sqrt{\frac{3}{27}}$		$\sqrt{\frac{4}{27}}$	$\sqrt{\frac{6}{27}}$	$-\sqrt{\frac{5}{27}}$	$-\sqrt{\frac{9}{27}}$
	17*			1													-1	
	71								1									-1

Table 5a (*cont.*)

$[\widetilde{332}]$ (012)	[$\widetilde{33}$] (002) 1113	3313	1131	3331	[$\widetilde{222}$] (030) 13* 11	3111	3511	53* 11	13* 33	3133	3533	53* 33	17* 33	7133
13	$-\sqrt{\frac{3}{4}}$			$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{3}{12}}$					$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{8}{12}}$			
31		$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$			$\sqrt{\frac{3}{12}}$			$\sqrt{\frac{1}{12}}$			$-\sqrt{\frac{8}{12}}$		
33		$\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$					$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{2}{6}}$	$\sqrt{\frac{2}{6}}$		
35		-1					$\sqrt{\frac{15}{150}}$		$\sqrt{\frac{2}{150}}$		$-\sqrt{\frac{45}{150}}$	$\sqrt{\frac{25}{150}}$	$\sqrt{\frac{63}{150}}$	
53				1				$-\sqrt{\frac{15}{150}}$		$-\sqrt{\frac{2}{150}}$	$-\sqrt{\frac{25}{150}}$	$\sqrt{\frac{45}{150}}$		$-\sqrt{\frac{63}{150}}$

(012)	[$\widetilde{321}$] (111) 13* 13	3113	$33_{\text{I}}13$	$33^*_{\text{II}}13$	15* 13	5113	3513	53* 13	13* 31	3131	$33_{\text{I}}31$	$33^*_{\text{II}}31$	15* 31	5131	3531	53* 31
13	$\sqrt{\frac{15}{64}}$				$-\sqrt{\frac{15}{64}}$						$-\sqrt{\frac{2}{64}}$	$-\sqrt{\frac{32}{64}}$				
31			$\sqrt{\frac{32}{64}}$	$\sqrt{\frac{2}{64}}$						$-\sqrt{\frac{15}{64}}$				$\sqrt{\frac{15}{64}}$		
33		$\sqrt{\frac{20}{128}}$	$-\sqrt{\frac{2}{128}}$	$-\sqrt{\frac{32}{128}}$			$-\sqrt{\frac{10}{128}}$		$-\sqrt{\frac{20}{128}}$		$\sqrt{\frac{32}{128}}$	$\sqrt{\frac{2}{128}}$				$\sqrt{\frac{10}{128}}$
35			$\sqrt{\frac{2}{64}}$	$-\sqrt{\frac{8}{64}}$			$-\sqrt{\frac{18}{64}}$						$-\sqrt{\frac{12}{64}}$		$\sqrt{\frac{24}{64}}$	
53						$\sqrt{\frac{12}{64}}$		$-\sqrt{\frac{24}{64}}$			$\sqrt{\frac{8}{64}}$	$-\sqrt{\frac{2}{64}}$				$\sqrt{\frac{18}{64}}$

(012)	13* 11	3111	$33_{\text{I}}11$	$33^*_{\text{II}}11$	3511	53* 11	13* 33	3133	$33_{\text{I}}33$	$33^*_{\text{II}}33$	15* 33	5133	3533	53* 33
13	$\sqrt{\frac{60}{384}}$							$-\sqrt{\frac{80}{384}}$	$-\sqrt{\frac{162}{384}}$	$-\sqrt{\frac{72}{384}}$			$-\sqrt{\frac{10}{384}}$	
31		$-\sqrt{\frac{60}{384}}$					$\sqrt{\frac{80}{384}}$		$\sqrt{\frac{72}{384}}$	$\sqrt{\frac{162}{384}}$				$\sqrt{\frac{10}{384}}$
33			$\sqrt{\frac{18}{192}}$	$-\sqrt{\frac{18}{192}}$			$\sqrt{\frac{5}{192}}$	$-\sqrt{\frac{5}{192}}$	$-\sqrt{\frac{18}{192}}$	$\sqrt{\frac{18}{192}}$	$\sqrt{\frac{45}{192}}$	$-\sqrt{\frac{45}{192}}$	$-\sqrt{\frac{10}{192}}$	$\sqrt{\frac{10}{192}}$
35					$-\sqrt{\frac{6}{96}}$		$-\sqrt{\frac{5}{96}}$		$\sqrt{\frac{18}{96}}$	·	$-\sqrt{\frac{9}{96}}$		$\sqrt{\frac{18}{96}}$	$\sqrt{\frac{40}{96}}$
53						$\sqrt{\frac{6}{96}}$		$\sqrt{\frac{5}{96}}$	·	$-\sqrt{\frac{18}{96}}$		$\sqrt{\frac{9}{96}}$	$-\sqrt{\frac{40}{96}}$	$-\sqrt{\frac{18}{96}}$

TABLE 5b. $\langle \gamma^n | \gamma^{n-2}\gamma^2 \rangle$ CHARGE-SPIN COEFFICIENTS (n ODD)

$[\tilde{3}], [\widetilde{43}], [\widetilde{443}]$		$[1], [\widetilde{41}], [\widetilde{441}]$ (100) 2213	2231	$[\widetilde{32}], [\widetilde{432}]$ (011) 2213	2413	2231	4231	2211	2233	2433	4233	$[\widetilde{333}]$ (003) 2211	2233	4433
(001)	22	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{8}{10}}$

$[\widetilde{111}]$		$[1]$ (100) 2211	2233
(300)	22	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
	44		1

$[\widetilde{21}], [\widetilde{421}]$		$[1], [\widetilde{41}]$ (100) 2213	2231	2211	2233	$[\widetilde{311}]$ (201) 2213	2413	4213	4413	2231	2431	4231	4431	2211	2411	4211	2233	2433	4233	4433
	22	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{4}{18}}$	$\sqrt{\frac{5}{18}}$			$-\sqrt{\frac{4}{18}}$		$\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{9}{108}}$	·		$-\sqrt{\frac{49}{108}}$	$-\sqrt{\frac{5}{108}}$	$-\sqrt{\frac{5}{108}}$	$-\sqrt{\frac{40}{108}}$
(110)	24	-1			-1	$-\sqrt{\frac{2}{72}}$	$-\sqrt{\frac{25}{72}}$				$\sqrt{\frac{5}{72}}$		$\sqrt{\frac{40}{72}}$		$-\sqrt{\frac{45}{432}}$		$\sqrt{\frac{2}{432}}$	$\sqrt{\frac{25}{432}}$	$\sqrt{\frac{160}{432}}$	$\sqrt{\frac{200}{432}}$
	42		-1		-1			$\sqrt{\frac{5}{72}}$	$\sqrt{\frac{40}{72}}$	$-\sqrt{\frac{2}{72}}$		$-\sqrt{\frac{25}{72}}$				$-\sqrt{\frac{45}{432}}$	$\sqrt{\frac{2}{432}}$	$\sqrt{\frac{160}{432}}$	$\sqrt{\frac{25}{432}}$	$\sqrt{\frac{200}{432}}$

		$[\widetilde{32}]$ (011) 2213	2413	4213	2231	2431	4231	2211	2411	4211	2233	2433	4233
	22	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$		$\sqrt{\frac{1}{4}}$		$-\sqrt{\frac{1}{4}}$				·	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
(110)	24	$-\sqrt{\frac{2}{16}}$	$\sqrt{\frac{5}{16}}$			$\sqrt{\frac{9}{16}}$			$\sqrt{\frac{1}{8}}$		$-\sqrt{\frac{2}{8}}$	$\sqrt{\frac{5}{8}}$	·
	42			$-\sqrt{\frac{9}{16}}$	$\sqrt{\frac{2}{16}}$		$-\sqrt{\frac{5}{16}}$			$-\sqrt{\frac{1}{8}}$	$\sqrt{\frac{2}{8}}$		$-\sqrt{\frac{5}{8}}$

		$[\widetilde{221}]$ (120) 2213	2413	4213	4413	2613	2231	2431	4231	4431	6231
	22	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{5}{12}}$				$\sqrt{\frac{1}{12}}$		$\sqrt{\frac{5}{12}}$		
(110)	24	$\sqrt{\frac{2}{48}}$	$-\sqrt{\frac{1}{48}}$			$\sqrt{\frac{24}{48}}$		$\sqrt{\frac{5}{48}}$		$\sqrt{\frac{16}{48}}$	
	42			$-\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{16}{48}}$		$-\sqrt{\frac{2}{48}}$		$\sqrt{\frac{1}{48}}$		$-\sqrt{\frac{24}{48}}$

$[\widetilde{41}], [\widetilde{441}]$		$[\tilde{3}], [\widetilde{43}]$ (001) 2213	2231	2211	2233	$[\widetilde{21}], [\widetilde{421}]$ (110) 2213	2413	2231	4231	$[\widetilde{331}]$ (102) 2211	2233	2433	4233	4433
(100)	22	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{9}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{4}{10}}$	$-\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{4}{10}}$	$-\sqrt{\frac{9}{90}}$	$\sqrt{\frac{1}{90}}$	$-\sqrt{\frac{20}{90}}$	$-\sqrt{\frac{20}{90}}$	$\sqrt{\frac{40}{90}}$

Table 5*b* (*cont.*)

$[\widetilde{32}]$, $[\widetilde{432}]$		$[\tilde{3}]$, $[\widetilde{43}]$ (001) 2213	2231	$[\widetilde{322}]$ (021) 2213	2413	4213	4413	2613	2231	2431	4231	4431	6231
	22	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{5}{12}}$				$-\sqrt{\frac{1}{12}}$		$-\sqrt{\frac{5}{12}}$		
(011)	24	-1		$\sqrt{\frac{2}{48}}$	$-\sqrt{\frac{1}{48}}$			$\sqrt{\frac{24}{48}}$		$-\sqrt{\frac{5}{48}}$		$-\sqrt{\frac{16}{48}}$	
	42		1			$-\sqrt{\frac{5}{48}}$	$-\sqrt{\frac{16}{48}}$		$\sqrt{\frac{2}{48}}$		$-\sqrt{\frac{1}{48}}$		$\sqrt{\frac{24}{48}}$

		2211	2411	4211	2233	2433	4233	4433	2633	6233
	22	$-\sqrt{\frac{9}{108}}$			$-\sqrt{\frac{25}{108}}$	$\sqrt{\frac{5}{108}}$	$\sqrt{\frac{5}{108}}$	$-\sqrt{\frac{64}{108}}$		
(011)	24		$-\sqrt{\frac{45}{432}}$		$-\sqrt{\frac{2}{432}}$	$\sqrt{\frac{49}{432}}$	$\sqrt{\frac{40}{432}}$	$-\sqrt{\frac{80}{432}}$	$\sqrt{\frac{216}{432}}$	
	42			$-\sqrt{\frac{45}{432}}$	$-\sqrt{\frac{2}{432}}$	$\sqrt{\frac{40}{432}}$	$\sqrt{\frac{49}{432}}$	$-\sqrt{\frac{80}{432}}$		$\sqrt{\frac{216}{432}}$

		$[\widetilde{21}]$, $[\widetilde{421}]$ (110) 2213	2413	4213	2231	2431	4231	2211	2411	4211	2233	2433	4233
	22	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$		$-\sqrt{\frac{1}{4}}$		$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$			$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$
(011)	24	$-\sqrt{\frac{2}{16}}$	$\sqrt{\frac{5}{16}}$			$-\sqrt{\frac{9}{16}}$			$\sqrt{\frac{1}{16}}$		$-\sqrt{\frac{2}{16}}$	$-\sqrt{\frac{5}{16}}$	$\sqrt{\frac{8}{16}}$
	42			$-\sqrt{\frac{9}{16}}$	$-\sqrt{\frac{2}{16}}$		$\sqrt{\frac{5}{16}}$			$\sqrt{\frac{1}{16}}$	$-\sqrt{\frac{2}{16}}$	$\sqrt{\frac{8}{16}}$	$-\sqrt{\frac{5}{16}}$

		$[\widetilde{331}]$ (102) 2213	2413	4213	4413	2231	2431	4231	4431	2211	2411	4211	2233	2433	4233	4433
	22	$-\sqrt{\frac{4}{18}}$	$\sqrt{\frac{5}{18}}$			$\sqrt{\frac{4}{18}}$		$-\sqrt{\frac{5}{18}}$					·	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
(011)	24	$-\sqrt{\frac{2}{72}}$	$-\sqrt{\frac{25}{72}}$				$-\sqrt{\frac{5}{72}}$		$-\sqrt{\frac{40}{72}}$		$-\sqrt{\frac{9}{72}}$		$\sqrt{\frac{10}{72}}$	$-\sqrt{\frac{5}{72}}$	$\sqrt{\frac{8}{72}}$	$-\sqrt{\frac{40}{72}}$
	42			$\sqrt{\frac{5}{72}}$	$\sqrt{\frac{40}{72}}$	$\sqrt{\frac{2}{72}}$		$\sqrt{\frac{25}{72}}$				$\sqrt{\frac{9}{72}}$	$-\sqrt{\frac{10}{72}}$	$-\sqrt{\frac{8}{72}}$	$\sqrt{\frac{5}{72}}$	$\sqrt{\frac{40}{72}}$

$[\widetilde{311}]$		$[\tilde{3}]$ (001) 2211	2233	$[\widetilde{21}]$ (110) 2213	2413	4213	2231	2431	4231	2211	2411	4211	2233	2433	4233	$[\widetilde{111}]$ (300) 2213	4413	2231	4431
	22	$-\sqrt{\frac{9}{10}}$	$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$		$\sqrt{\frac{4}{10}}$		$-\sqrt{\frac{1}{10}}$	·			·	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$	
(201)	24		1	$-\sqrt{\frac{2}{8}}$	$-\sqrt{\frac{5}{8}}$			$-\sqrt{\frac{1}{8}}$			$-\sqrt{\frac{9}{40}}$		$-\sqrt{\frac{18}{40}}$	$-\sqrt{\frac{5}{40}}$	$-\sqrt{\frac{8}{40}}$	$-\sqrt{\frac{1}{5}}$			$-\sqrt{\frac{4}{5}}$
	42		1			$\sqrt{\frac{1}{8}}$	$\sqrt{\frac{2}{8}}$		$\sqrt{\frac{5}{8}}$			$\sqrt{\frac{9}{40}}$	$\sqrt{\frac{18}{40}}$	$\sqrt{\frac{8}{40}}$	$\sqrt{\frac{5}{40}}$		$\sqrt{\frac{4}{5}}$	$\sqrt{\frac{1}{5}}$	
	44		1			$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$					·	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$

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Table 5*b* (*cont.*)

$\widetilde{[221]}$		$\widetilde{[21]}$ (110)												$\widetilde{[111]}$ (300)			
		22 13	24 13	42 13	22 31	24 31	42 31	22 11	24 11	42 11	22 33	24 33	42 33	22 11	44 11	22 33	44 33
	22	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$		$-\sqrt{\frac{1}{4}}$		$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{9}{36}}$			$-\sqrt{\frac{25}{36}}$	$\sqrt{\frac{1}{36}}$	$\sqrt{\frac{1}{36}}$	$-\sqrt{\frac{9}{18}}$		$\sqrt{\frac{1}{18}}$	$\sqrt{\frac{8}{18}}$
	24	$\sqrt{\frac{10}{16}}$	$-\sqrt{\frac{1}{16}}$			$-\sqrt{\frac{5}{16}}$			$-\sqrt{\frac{45}{144}}$		$-\sqrt{\frac{10}{144}}$	$\sqrt{\frac{49}{144}}$	$\sqrt{\frac{40}{144}}$			$\sqrt{\frac{5}{9}}$	$-\sqrt{\frac{4}{9}}$
(120)	42			$-\sqrt{\frac{5}{16}}$	$\sqrt{\frac{10}{16}}$		$-\sqrt{\frac{1}{16}}$			$-\sqrt{\frac{45}{144}}$	$-\sqrt{\frac{10}{144}}$	$\sqrt{\frac{40}{144}}$	$\sqrt{\frac{49}{144}}$			$\sqrt{\frac{5}{9}}$	$-\sqrt{\frac{4}{9}}$
	44			$\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$					$-\sqrt{\frac{8}{18}}$	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{9}{36}}$	$-\sqrt{\frac{2}{36}}$	$\sqrt{\frac{25}{36}}$
	26		-1									-1					-1
	62						-1						-1				-1

$\widetilde{[331]}$		$\widetilde{[32]}$ (011)											
		22 13	24 13	42 13	22 31	24 31	42 31	22 11	24 11	42 11	22 33	24 33	42 33
	22	$-\sqrt{\frac{4}{10}}$	$\sqrt{\frac{1}{10}}$		$-\sqrt{\frac{4}{10}}$		$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{9}{60}}$			$-\sqrt{\frac{49}{60}}$	$-\sqrt{\frac{1}{60}}$	$-\sqrt{\frac{1}{60}}$
(102)	24	$-\sqrt{\frac{2}{8}}$	$-\sqrt{\frac{5}{8}}$			$\sqrt{\frac{1}{8}}$			$-\sqrt{\frac{9}{48}}$		$\sqrt{\frac{2}{48}}$	$\sqrt{\frac{5}{48}}$	$\sqrt{\frac{32}{48}}$
	42			$\sqrt{\frac{1}{8}}$	$-\sqrt{\frac{2}{8}}$		$-\sqrt{\frac{5}{8}}$			$-\sqrt{\frac{9}{48}}$	$\sqrt{\frac{2}{48}}$	$\sqrt{\frac{32}{48}}$	$\sqrt{\frac{5}{48}}$
	44			$-\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$					$-\sqrt{\frac{2}{12}}$	$-\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{5}{12}}$

		$\widetilde{[311]}$ (201)							
		22 13	24 13	42 13	44 13	22 31	24 31	42 31	44 31
	22	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{5}{12}}$			$\sqrt{\frac{1}{12}}$		$\sqrt{\frac{5}{12}}$	
(102)	24	$\sqrt{\frac{10}{48}}$	$-\sqrt{\frac{5}{48}}$				$\sqrt{\frac{25}{48}}$		$-\sqrt{\frac{8}{48}}$
	42			$-\sqrt{\frac{25}{48}}$	$\sqrt{\frac{8}{48}}$	$-\sqrt{\frac{10}{48}}$		$\sqrt{\frac{5}{48}}$	
	44			$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{5}{12}}$		$\sqrt{\frac{1}{12}}$		$-\sqrt{\frac{5}{12}}$

		$\widetilde{[221]}$ (120)									
		22 11	24 11	42 11	44 11	22 33	24 33	42 33	44 33	26 33	62 33
	22	$\sqrt{\frac{9}{36}}$				$-\sqrt{\frac{1}{36}}$	$-\sqrt{\frac{5}{36}}$	$-\sqrt{\frac{5}{36}}$	$-\sqrt{\frac{16}{36}}$		
(102)	24		$-\sqrt{\frac{45}{720}}$			$\sqrt{\frac{50}{720}}$	$\sqrt{\frac{169}{720}}$	$-\sqrt{\frac{160}{720}}$	$-\sqrt{\frac{80}{720}}$	$-\sqrt{\frac{216}{720}}$	
	42			$-\sqrt{\frac{45}{720}}$		$\sqrt{\frac{50}{720}}$	$-\sqrt{\frac{160}{720}}$	$\sqrt{\frac{169}{720}}$	$-\sqrt{\frac{80}{720}}$		$-\sqrt{\frac{216}{720}}$
	44				$\sqrt{\frac{18}{180}}$	$\sqrt{\frac{2}{180}}$	$-\sqrt{\frac{1}{180}}$	$-\sqrt{\frac{1}{180}}$	$-\sqrt{\frac{50}{180}}$	$-\sqrt{\frac{54}{180}}$	$-\sqrt{\frac{54}{180}}$

Table 5*b* (*cont.*)

$[\widetilde{322}]$		$[\widetilde{32}]$ (011)						$[\widetilde{311}]$ (201)							
		2213	2413	4213	2231	2431	4231	2211	2411	4211	4411	2233	2433	4233	4433
(021)	22	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$		$\sqrt{\frac{1}{4}}$		$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{45}{108}}$				$-\sqrt{\frac{5}{108}}$	$-\sqrt{\frac{25}{108}}$	$-\sqrt{\frac{25}{108}}$	$\sqrt{\frac{8}{108}}$
	24	$\sqrt{\frac{10}{16}}$	$-\sqrt{\frac{1}{16}}$			$\sqrt{\frac{5}{16}}$			$-\sqrt{\frac{45}{432}}$			$\sqrt{\frac{50}{432}}$	$\sqrt{\frac{169}{432}}$	$-\sqrt{\frac{160}{432}}$	$\sqrt{\frac{8}{432}}$
	42			$-\sqrt{\frac{5}{16}}$	$-\sqrt{\frac{10}{16}}$		$\sqrt{\frac{1}{16}}$			$-\sqrt{\frac{45}{432}}$		$\sqrt{\frac{50}{432}}$	$-\sqrt{\frac{160}{432}}$	$\sqrt{\frac{169}{432}}$	$\sqrt{\frac{8}{432}}$
	44			$\sqrt{\frac{1}{2}}$		$-\sqrt{\frac{1}{2}}$					$\sqrt{\frac{9}{54}}$	$-\sqrt{\frac{10}{54}}$	$\sqrt{\frac{5}{54}}$	$\sqrt{\frac{5}{54}}$	$-\sqrt{\frac{25}{54}}$
	26		-1										$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{2}{3}}$
	62						1							$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$

		$[\widetilde{221}]$ (120)											
		2213	2413	4213	4413	2613	6213	2231	2431	4231	4431	2631	6231
(021)	22	$-\sqrt{\frac{5}{12}}$	$\sqrt{\frac{1}{12}}$					$\sqrt{\frac{5}{12}}$		$-\sqrt{\frac{1}{12}}$			
	24	$-\sqrt{\frac{50}{1200}}$	$\sqrt{\frac{289}{1200}}$			$\sqrt{\frac{216}{1200}}$			$\sqrt{\frac{245}{1200}}$		$-\sqrt{\frac{400}{1200}}$		
	42			$-\sqrt{\frac{245}{1200}}$	$\sqrt{\frac{400}{1200}}$			$\sqrt{\frac{50}{1200}}$		$-\sqrt{\frac{289}{1200}}$			$-\sqrt{\frac{216}{1200}}$
	44			$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{2}{6}}$				$\sqrt{\frac{1}{6}}$		$-\sqrt{\frac{2}{6}}$		
	26		$-\sqrt{\frac{3}{25}}$			$-\sqrt{\frac{7}{25}}$						$\sqrt{\frac{15}{25}}$	
	62						$-\sqrt{\frac{15}{25}}$			$\sqrt{\frac{3}{25}}$			$\sqrt{\frac{7}{25}}$

		2211	2411	4211	4411	2611	6211	2233	2433	4233	4433	2633	6233
(021)	22							·	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	·		
	24		$\sqrt{\frac{45}{200}}$					$\sqrt{\frac{50}{200}}$	$-\sqrt{\frac{1}{200}}$	·	$-\sqrt{\frac{80}{200}}$	$\sqrt{\frac{24}{200}}$	
	42			$-\sqrt{\frac{45}{120}}$				$-\sqrt{\frac{50}{200}}$	·	$\sqrt{\frac{1}{200}}$	$\sqrt{\frac{80}{200}}$		$-\sqrt{\frac{24}{200}}$
	44							·	$\sqrt{\frac{2}{10}}$	$-\sqrt{\frac{2}{10}}$	·	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{3}{10}}$
	26					$-\sqrt{\frac{5}{50}}$			$-\sqrt{\frac{4}{50}}$		$\sqrt{\frac{20}{50}}$	$\sqrt{\frac{21}{50}}$	
	62						$\sqrt{\frac{5}{50}}$			$\sqrt{\frac{4}{50}}$	$-\sqrt{\frac{20}{50}}$		$-\sqrt{\frac{21}{50}}$

$[\widetilde{333}]$		$[\widetilde{331}]$ (102)								$[\widetilde{322}]$ (021)							
		2213	2413	4213	4413	2231	2431	4231	4431	2211	4411	2233	2433	4233	4433	2633	6233
(003)	22	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{4}{18}}$			$-\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{4}{18}}$		$-\sqrt{\frac{9}{54}}$		$\sqrt{\frac{1}{54}}$	$-\sqrt{\frac{20}{54}}$	$-\sqrt{\frac{20}{54}}$	$-\sqrt{\frac{4}{54}}$		
	44			$-\sqrt{\frac{4}{18}}$	$-\sqrt{\frac{5}{18}}$		$\sqrt{\frac{4}{18}}$		$\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{9}{108}}$	$\sqrt{\frac{4}{108}}$	$\sqrt{\frac{8}{108}}$	$\sqrt{\frac{8}{108}}$	$\sqrt{\frac{25}{108}}$	$-\sqrt{\frac{27}{108}}$	$-\sqrt{\frac{27}{108}}$

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TABLE 6. NON-VANISHING MATRIX ELEMENTS OF $X = \frac{15}{16}\sum_{i<j}\{(u_i^{(1)}\cdot u_j^{(1)}) - (u_i^{(2)}\cdot u_j^{(2)})\}\{(\vec{\tau}_i\cdot\vec{\tau}_j) - (\vec{\sigma}_i\cdot\vec{\sigma}_j)\}$

(Reproducing, with changed phases, a table previously published by Racah.)

partition		charge-spin factor									orbital factor								p^{12-n}	
	⟨[f] | [f′]⟩	⟨(g) | (g′)⟩	11	13	31	33	33_{I}	33_{II}	15	51	⟨(g) | (g′)⟩	S	P	D	D_{I}	D_{II}	F	G	⟨[f] | [f′]⟩	
p^2	⟨2 | 2⟩	⟨010 | 010⟩		−1	1						⟨20 | 20⟩	$-\frac{5}{2}$		$\frac{1}{2}$					⟨442 | 442⟩	p^{10}
p^4	⟨211 | 31⟩	⟨210 | 101⟩		$-\sqrt{3}$	$-\sqrt{3}$	2*					⟨10 | 21⟩		$\sqrt{\frac{5}{4}}$						⟨332 | 431⟩	p^8
	⟨22 | 22⟩	⟨020 | 020⟩	0			0			−1	1	⟨02 | 02⟩	$-\frac{5}{2}$		$\frac{1}{2}$					⟨422 | 422⟩	
	⟨22 | 31⟩	⟨020 | 101⟩				1					⟨02 | 21⟩			$\sqrt{3}$					⟨422 | 431⟩	
	⟨22 | 4⟩	⟨020 | 000⟩	1								⟨02 | 40⟩	$-\sqrt{40}$		$\sqrt{7}$					⟨422 | 44⟩	
	⟨31 | 31⟩	⟨101 | 101⟩		−1	1	0					⟨21 | 21⟩		3	$-\frac{5}{2}$			$\frac{1}{2}$		⟨431 | 431⟩	
p^6	⟨222 | 42⟩	⟨030 | 010⟩		−1	−1						⟨00 | 22⟩	$\sqrt{\frac{75}{4}}$								
	⟨321 | 321⟩	⟨111 | 111⟩		$-\frac{3}{5}$	$\frac{3}{5}$		$-\frac{2}{3}$	$\frac{2}{3}$	−1	1	⟨11 | 11⟩		$\frac{5}{2}$	$-\frac{3}{2}$						
	⟨321 | 411⟩	⟨111 | 200⟩					1	1			⟨11 | 30⟩		$\sqrt{\frac{40}{9}}$							
	⟨321 | 33⟩	⟨111 | 002⟩					1	−1			⟨11 | 03⟩		$\sqrt{\frac{40}{9}}$							
	⟨321 | 42⟩	⟨111 | 010⟩		1	1						⟨11 | 22⟩				·	$-\sqrt{\frac{336}{25}}$				
	⟨411 | 33⟩	⟨200 | 002⟩	1			$-\frac{1}{3}$					⟨30 | 03⟩		−2				3			
	⟨42 | 42⟩	⟨010 | 010⟩		−1	1						⟨22 | 22⟩	−3			$-\frac{3}{2}$	$\frac{9}{10}$	$\frac{3}{2}$	$-\frac{1}{2}$		

partition		charge-spin factor					orbital factor					p^{12-n}	
	⟨[f] | [f′]⟩	⟨(g) | (g′)⟩	22	24	42	44	⟨(g) | (g′)⟩	S	P	D	F	⟨[f] | [f′]⟩	
p^3	⟨21 | 21⟩	⟨110 | 110⟩	0	$\frac{1}{2}$	$-\frac{1}{2}$		⟨11 | 11⟩		$\frac{5}{2}$	$-\frac{3}{2}$		⟨432 | 432⟩	p^9
	⟨21 | 3⟩	⟨110 | 001⟩	$\frac{3}{2}$				⟨11 | 30⟩		$\sqrt{\frac{40}{9}}$			⟨432 | 441⟩	
p^5	⟨221 | 32⟩	⟨120 | 011⟩	0	$\sqrt{\frac{15}{4}}$	$-\sqrt{\frac{15}{4}}$		⟨01 | 12⟩		$\sqrt{\frac{5}{4}}$			⟨322 | 421⟩	p^7
	⟨221 | 41⟩	⟨120 | 100⟩	1				⟨01 | 31⟩		$\sqrt{15}$			⟨322 | 43⟩	
	⟨311 | 311⟩	⟨201 | 201⟩	0	$\frac{3}{2}$	$-\frac{3}{2}$	0	⟨20 | 20⟩	$-\frac{5}{2}$		$\frac{1}{2}$		⟨331 | 331⟩	
	⟨311 | 32⟩	⟨201 | 011⟩	$\sqrt{\frac{12}{5}}$	$-\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{3}{4}}$		⟨20 | 12⟩			$\sqrt{3}$		⟨331 | 421⟩	
	⟨32 | 32⟩	⟨011 | 011⟩	0	$\frac{1}{2}$	$-\frac{1}{2}$		⟨12 | 12⟩		3	$-\frac{5}{2}$	$\frac{1}{2}$	⟨421 | 421⟩	
	⟨32 | 41⟩	⟨011 | 100⟩	$\frac{3}{2}$				⟨12 | 31⟩		2	$\sqrt{\frac{56}{45}}$	$-\frac{4}{3}$	⟨421 | 43⟩	

* The star on the element 2 indicates that this becomes −2 for the p^8 configuration (with the standard (amended) phases of part A). The remaining matrix elements do not change sign in going over to the contragredient representations of p^{12-n}; for the latter the listed values of (g_1g_2) or $(g_1g_2g_3)$ for p^n have to be replaced by (g_2g_1) or $(g_3g_2g_1)$ respectively, similarly for (g'). There are no coupling terms between the [42] D_{I} and [42] D_{II} states nor between the $[321]^{33}_{\mathrm{I}}\,L$ and $[321]^{33}_{\mathrm{II}}\,L$ states.